SAPIENZA
UNIVERSITÀ DI ROMA

Functional identities of certain zeta-like functions associated to Drinfeld A-modules

Scuola Dottorale Vito Volterra
PhD in Mathematics (XXXVII cycle)

Giacomo Hermes Ferraro
ID number 1996686

Advisor
Prof. Federico Pellarin

Coordinator
Prof. Emanuele Spadaro

Academic Year 2023/24

Reviewers:

Prof. Matt Papanikolas
Prof. Tuan Ngo Dac

Functional identities of certain zeta-like functions associated to Drinfeld A-modules
Sapienza Università di Roma

This thesis has been typeset by LaTeX and the Sapthesis class.

Author's email: giacomohermes.ferraro@gmail.com

A mio fratello, i miei genitori e i miei nonni, che hanno sempre creduto in me e a cui devo tutto
A Frank, Furio, Gori e Casarin, la cui compagnia di questi otto e più anni è stata inestimabile
A Gennaro e Benedetta, sulla cui amicizia ho sempre potuto contare e sempre conterò
A Maria, che è la mia più grande fonte di felicità e di ispirazione

Chapter 1

Introduction

Zeta functions hold a place of particular significance in number theory. The foundational text is [Rie59] by Riemann, in which he proved that the function of a complex variable

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s},$$

defined for all s with $\Re(s) > 1$, can be extended to a meromorphic function on the complex plane, and that it satisfies the functional identity:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(s)\zeta(1-s),$$

where $\Gamma(s)$ is the Gamma function, a meromorphic function which, for $\Re(s) > 0$, is defined as

$$\Gamma(s) := \int_0^{\infty} x^{s-1} \exp(-x) \mathrm{d}x.$$

In the same article, he formulated arguably the most famous open conjecture in all of mathematics, namely that the zeros of $\zeta(s)$ different from the negative even integers lie on the line $s = \frac{1}{2} + it$ with $t \in \mathbb{R}$.

Over time, many different zeta functions have been defined to generalize Riemann's, usually satisfying some functional identity, with an associated conjecture generalizing Riemann's hypothesis. This is the case for Dedekind zeta functions, which generalize Riemann's zeta to arbitrary number fields, and Dirichlet L-series, which depend on the choice of a certain multiplicative character $\chi : \mathbb{Z} \to \mathbb{C}$.

In the world of finite characteristic, the theory of zeta functions has proven to be particularly rich. This theory was pioneered by Artin in his PhD thesis, which focused on quadratic function fields (see [Art24]); in the following decades, Hasse proved an analogue of Riemann's hypothesis for elliptic function fields ([Has36]), and Weil developed a comprehensive theory to prove this conjecture for arbitrary function fields with finite base field ([Wei48]).

At the same time, another line of research on curves over a finite field $\mathbb{F}_q$ was being developed by Carlitz, starting in [Car35]. He attached to the polynomial ring $\mathbb{F}_q[\theta]$ an $\mathbb{F}_q$-linear *exponential function* $\exp_C(z)$, whose domain and codomain consist in an algebraically closed complete normed field $\mathbb{C}_\infty$ extending the field of Laurent series $\mathbb{F}_q((\theta^{-1}))$ in the variable θ. If we interpret $\mathbb{F}_q[t]$ as the function field equivalent of $\mathbb{Z}$ (being the "simplest" principal ideal domain among $\mathbb{F}_q$-algebras), $\mathbb{C}_\infty$ is analogous to the field of complex numbers $\mathbb{C}$ and $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ to the classical exponential

$\exp : \mathbb{C} \to \mathbb{C}^\times$. As the latter induces the multiplicative $\mathbb{Z}$-module structure on its image $\mathbb{C}^\times$, Carlitz proved that $\exp_C(\mathbb{C}_\infty) = \mathbb{C}_\infty$ can be endowed with a natural action of $\mathbb{F}_q[t] = \mathbb{F}_q[\theta]$, now called the *Carlitz module*.

Carlitz was interested in the computation of the following zeta-like values:

$$\zeta_C((q-1)m) := \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ \text{monic}}} a^{-(q-1)m} \in \mathbb{C}_\infty,$$

where m is a positive integer. An explicit formula for classical zeta values at the even positive integers was known well before Riemann's work, due to Euler:

$$\zeta(2k) = -\frac{1}{2} \cdot \frac{B_{2k}}{(2k)!} \cdot (2\pi i)^{2k}$$

for all positive integers k, where the Bernoulli numbers B_{2k} appear in the generating function

$$\frac{z}{\exp(z) - 1} = \sum_{k \geq 0} \frac{B_k}{k!} z^k.$$

In [Car37] and [Car40] Carlitz proved the identity:

$$\zeta_C((q-1)m) = \frac{BC_{(q-1)m}}{\Pi((q-1)m)} \cdot \tilde{\pi}^{(q-1)m}$$

for all positive integers m, where $\Pi : \mathbb{Z} \to \mathbb{F}_q[\theta]$ is an analogue of the factorial, the Bernoulli–Carlitz numbers $BC_{(q-1)m} \in \mathbb{F}_q(\theta)$ appear in the generating function

$$\frac{z}{\exp_C(z)} = \sum_{k \geq 0} \frac{BC_k}{\Pi_k} z^k$$

and satisfy an analogue of the Von Staudt–Clausen theorem, and $\tilde{\pi} \in \mathbb{C}_\infty$ is an analogue of $2\pi i$ (which was proven to be transcendental over $\mathbb{F}_q(\theta)$ by Carlitz's student Wade [Wad41]).

Several decades later, in [Gos78], Goss was prompted by this wealth of analogies with the classical case to define the zeta function

$$\zeta_C(s) := \sum_{\substack{a \in \mathbb{F}_q[\theta] \\ \text{monic}}} a^{-s}$$

for all positive integers s; he later extended the domain of ζ_C to a large set of exponents S_∞, analogous to the complex plane ([Gos79]). Goss's dream—yet unrealized—was to find some automorphism of S_∞ analogous to the involution $s \leftrightarrow 1 - s$ on the complex plane, and to prove a functional identity of $\zeta_C(s)$ under its action.

Drinfeld modules ([Dri74]) and Anderson modules ([And86]) are generalizations of the Carlitz module, analogous respectively to the classical theories of elliptic curves and abelian varieties; in the following years, Goss developed a theory of L-functions in this generality ([Gos92]). Starting with Taelman, who studied the case of Drinfeld modules in [Tae12], and ending with the article [ANT22] by Anglès, Ngo Dac, and Tavares Ribeiro, in which they worked in the generality of Anderson modules, the special value of Goss L-functions at $s = 1$ was proven to be the product of a regulator and an algebraic term arising from a certain class module, in analogy with the class number formula attached to Dedekind L-functions.

At the same time, a new type of L-series attached to the affine curve $\mathrm{Spec}(\mathbb{F}_q[t]) = \mathbb{P}^1_{\mathbb{F}_q} \setminus \{\infty\}$ was defined by Pellarin in [Pel12] for positive integers s in the Tate algebra $\mathbb{C}_\infty\langle t\rangle$:

$$L(s) \coloneqq \sum_{\substack{a\in\mathbb{F}_q[t]\setminus\{0\}\\ \text{monic}}} \frac{a(t)}{a(\theta)^{-s}}.$$

He proved that the value $L(1)$ is a rigid-analytic function on $\mathbb{P}^1_{\mathbb{C}_\infty} \setminus \{\infty\}$ which can be used to interpolate the Carlitz zeta values $\zeta_C(q^k - 1)$ for all positive integers k, and found a functional identity relating $L(1)$ and the so-called Anderson–Thakur special function $\omega \in \mathbb{C}_\infty\langle t\rangle$. This identity bears a striking similarity with the evaluation of Riemann's classical functional identity at $s = 0$ (explored in more detail in the rest of the introduction), and suggests that Pellarin-type L-values are an alternative to Goss L-series as a function field analogue of classical L-series.

A Pellarin-type L-value can be attached to any pair (X, ∞), where X is a "nice" projective curve and $\infty \in X$ an $\mathbb{F}_q$-rational point. Interestingly, if we interpret it as a function from $X(\mathbb{C}_\infty) \setminus \infty$ to $\mathbb{C}_\infty$, it has an infinite number of trivial zeros indexed by the natural numbers and a (finite) set of nontrivial zeros; determining this nontrivial zeros is an important step to generalize Pellarin's identity to arbitrary function fields. The central chapters of this thesis are devoted to this objective, expanding previous work by Green and Papanikolas ([GP18]). An essential idea emerging from those chapters is that Pellarin-type L values derive their properties from being eigenvectors of certain operators: using this insight we are able to further generalize our main theorems to arbitrary Drinfeld modules, and to formulate several conjectures in the more general context of Anderson modules.

1.1 The Carlitz module

Let $\mathbb{F}_q$ be the finite field with q elements. The Carlitz module over the field of rational functions $\mathbb{F}_q(\theta)$ is a functor from the $\mathbb{F}_q[\theta]$-algebras to the $\mathbb{F}_q[t]$-modules

$$C : \mathbb{F}_q[\theta] - \mathbf{Alg} \to \mathbb{F}_q[t] - \mathbf{Mod},$$

which is meant to be a finite characteristic analogue to the multiplicative group scheme

$$\mathbb{G}_m : \mathbb{Z} - \mathbf{Alg} \to \mathbb{Z} - \mathbf{Mod}.$$

For any $\mathbb{F}_q[\theta]$-algebra S, $C(S)$ is defined as the $\mathbb{F}_q$-vector space S endowed with an $\mathbb{F}_q[t]$-module structure uniquely determined by the following action of t:

$$C_t(s) := s^q + \theta s \text{ for all } s \in S.$$

Let's denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of $\mathbb{F}_q((\theta^{-1}))$: this is a complete algebraically closed field analogous to the field of complex numbers $\mathbb{C}$. Analogously to the classical case, there is a surjective map of A-modules $\exp_C : \mathbb{C}_\infty \to C(\mathbb{C}_\infty)$, with kernel $\tilde{\pi}A \subseteq \mathbb{C}_\infty$ for a certain $\tilde{\pi} \in \mathbb{C}_\infty^\times$.

1.1.1 Gauss–Thakur sums

Let's recall the notion of a Gauss sum.

Definition (Gauss)**.** Fix an integer $n \in \mathbb{Z} \setminus \{0, \pm 1\}$. Denote by $\mathbb{G}_m[n] \subseteq \mathbb{G}_m$ the subfunctor of n-torsion and let $\mu_n \coloneqq \mathbb{G}_m[n](\mathbb{C})$ be the set of n-th roots of unity. Fix a multiplicative character

$$\rho : {\mathbb{Z}}/{n\mathbb{Z}}^{\times} \to \mathbb{C}^{\times}$$

and an isomorphism of $\mathbb{Z}$-modules

$$\zeta : {\mathbb{Z}}/{n\mathbb{Z}} \xrightarrow{\sim} \mu_n.$$

The Gauss sum relative to ρ and ζ is defined as follows:

$$\mathcal{G}(\rho, \zeta) = \sum_{m \in {\mathbb{Z}}/{n\mathbb{Z}}^{\times}} \rho(m)^{-1} \zeta(m) \in \mathbb{C}.$$

Remark. We slightly deviated from historical notation by using $\rho(m)^{-1}$ instead of $\rho(m)$ in the summation.

If we fix a prime number $p \in \mathbb{Z}$ and an isomorphism $\zeta : {\mathbb{Z}}/{p\mathbb{Z}} \xrightarrow{\sim} \mu_p$, the $p-1$ Gauss sums

$$\left\{ \mathcal{G}(\rho, \zeta) | \rho : {\mathbb{Z}}/{p\mathbb{Z}}^{\times} \to \mathbb{C}^{\times} \right\} \subseteq \mathbb{Q}(\mu_p, \mu_{p-1})$$

form a full set of simultaneous eigenvectors for the Galois group $\operatorname{Gal}\left({\mathbb{Q}(\mu_p, \mu_{p-1})}/{\mathbb{Q}(\mu_{p-1})} \right)$. This makes them an important tool for the study of cyclotomic extensions of $\mathbb{Q}$.

In the seminal paper [Tha88], Thakur developed an analogous object to Gauss sums in the context of the Carlitz module.

Definition (Thakur)**.** Fix an element $b \in \mathbb{F}_q[t] \setminus \mathbb{F}_q$. Denote by $C[b] \subseteq C$ the subfunctor of b-torsion and let $\nu_b \coloneqq C[b](\mathbb{C}_\infty)$ be the set of roots of C_b. Fix a multiplicative character

$$\chi : {\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]}^{\times} \to \mathbb{C}_\infty^{\times}$$

and an isomorphism of $\mathbb{F}_q[t]$-modules

$$\xi : {\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]} \xrightarrow{\sim} \nu_b.$$

The Gauss–Thakur sum relative to χ and ξ is defined as follows:

$$\mathfrak{g}(\chi, \xi) = \sum_{a \in {\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]}^{\times}} \chi(a)^{-1} \xi(a) \in \mathbb{C}_\infty,$$

where we identified $C(\mathbb{C}_\infty)$ with $\mathbb{C}_\infty$.

Given an element $b \in \mathbb{F}_q[t] \setminus \mathbb{F}_q$, the map sending $a \in \mathbb{F}_q[t]$ to the endomorphism $C_a : \nu_b \to \nu_b$ induces an isomorphism of $\mathbb{F}_q[t]$-modules between ${\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]}$ and $\operatorname{Gal}\left({\overline{\mathbb{F}_q}(\theta)[\nu_b]}/{\overline{\mathbb{F}_q}(\theta)} \right)$.

Remark. Similarly to Gauss sums, if we fix a prime element $b \in \mathbb{F}_q[t]$ and an isomorphism $\xi : {\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]} \xrightarrow{\sim} \nu_b$, the nonzero Gauss–Thakur sums in

$$\left\{ \mathfrak{g}(\chi, \xi) | \chi : {\mathbb{F}_q[t]}/{b\mathbb{F}_q[t]}^{\times} \to \mathbb{C}_\infty^{\times} \right\} \subseteq \overline{\mathbb{F}_q}(\theta)[\nu_b]$$

form a full set of simultaneous eigenvectors for the Galois group $\operatorname{Gal}\left({\overline{\mathbb{F}_q}(\theta)[\nu_b]}/{\overline{\mathbb{F}_q}(\theta)} \right)$.

The theorem of Kronecker–Weber asserts that the maximal abelian field extensions of $\mathbb{Q}$ is the colimit of the cyclotomic extensions. In his paper [Hay74], Hayes proves an analogous statement for the function field $\mathbb{F}_q(\theta)$. If we denote by $C(\mathbb{C}_\infty)^{tors}$ the torsion submodule of the $\mathbb{F}_q[t]$-module $C(\mathbb{C}_\infty)$, i.e. $\bigcup_{b\in\mathbb{F}_q[t]} \nu_b$, we have the following.

Theorem (Hayes)**.** *Let's identify $C(\mathbb{C}_\infty)$ with $\mathbb{C}_\infty$. The field $\overline{\mathbb{F}_q}(\theta)[C(\mathbb{C}_\infty)^{tors}]$ is the maximal abelian extension of $\mathbb{F}_q(\theta)$ which is tamely ramified at ∞.*

1.1.2 Anderson–Thakur special function

In their seminal paper [AT90], Anderson and Thakur introduced the following series in $\mathbb{C}_\infty\langle t\rangle$ to study the Carlitz module.

Definition. The Anderson–Thakur special function is defined as follows:

$$\omega := (-\theta)^{\frac{1}{q-1}} \prod_{i\geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1}.$$

Remark. If we denote by $\tau : \mathbb{C}_\infty[[t]] \to \mathbb{C}_\infty[[t]]$ the map sending a series $\sum_i d_i t^i$ to $\sum_i d_i^q t^i$, we get that $\tau\omega = (t-\theta)\omega$. In other words, if we write $\omega = \sum_i c_i t^i$, we have the identities $c_i = C_\theta(c_{i+1})$ for all $i \geq 0$. Furthermore, from its definition, it's clear that ω can be evaluated at $t = \zeta$ for any $\zeta \in \overline{\mathbb{F}_q} \subseteq \mathbb{C}_\infty$.

Let's briefly explain the reason why in the last Remark of Subsection 1.1.1 we only considered nonzero Gauss–Thakur sums: it turns out that, for all $b \in \mathbb{F}_q[t] \setminus \mathbb{F}_q$, whenever a multiplicative character χ from ${}^{\mathbb{F}_q[t]}\!/\!_{b\mathbb{F}_q[t]}{}^{\times}$ to $\mathbb{C}_\infty^\times$ cannot be lifted to a morphism of $\mathbb{F}_q$-algebras ${}^{\mathbb{F}_q[t]}\!/\!_{b\mathbb{F}_q[t]} \to \mathbb{C}_\infty$, the associated Gauss–Thakur sum is zero (see [GM21][Prop. 4.8]). This suggests the existence of a "universal" Gauss–Thakur sum, independent from χ, whose specialization yields all possible nonzero Gauss–Thakur sums.

Anglès and Pellarin showed in [AP14] that the "universal" Gauss–Thakur sum is the special function of Anderson–Thakur.

Theorem ([AP14][Thm. 2.9])**.** *Let $\mathfrak{p} \in \mathbb{F}_q[t]$ be a nonzero prime and fix the isomorphism*

$$\xi : {}^{\mathbb{F}_q[t]}\!/\!_{\mathfrak{p}\mathbb{F}_q[t]} \cong \nu_\mathfrak{p}$$

sending $a \in \mathbb{F}_q[t]$ to $\exp_C\left(a\tilde{\pi}\frac{\mathfrak{p}'}{\mathfrak{p}}\right)$, where $\mathfrak{p}'$ denotes the derivative of $\mathfrak{p}$ in the variable t. For all algebra homomorphisms

$$\chi : \mathbb{F}_q[t] \to {}^{\mathbb{F}_q[t]}\!/\!_{\mathfrak{p}\mathbb{F}_q[t]} \to \mathbb{C}_\infty$$

we have the following identity:

$$\mathfrak{g}(\chi, \xi) = \chi(\omega),$$

where ω is the special function of Anderson–Thakur.

1.2 Pellarin's zeta value and its functional identity

The infinite series $\sum h^{-s} \in \mathbb{C}_\infty$, where we sum over all monic $h \in \mathbb{F}_q[\theta]$, is well defined for any positive integer s, and is the function field analogue of the Riemann zeta value at s. In a seminal paper ([Pel12]), Pellarin introduced the following series for any positive integer s:

$$L(s) := \sum_{\substack{h\in\mathbb{F}_q[t]\\ \text{monic}}} \frac{h(t)}{h(\theta)^s} \in \mathbb{C}_\infty[[t]].$$

For any $\mathbb{F}_q$-algebra homomorphism $\chi : \mathbb{F}_q[t] \to \overline{\mathbb{F}_q} \subseteq \mathbb{C}_\infty$, $L(s)$ can be evaluated at $\chi(t)$. The result $L(\chi, s) := \sum h^{-s}\chi(h)$ is analogue to a Dirichlet L-series, evaluated at s. In other words, the Pellarin zeta function interpolates certain Dirichlet-like series $L(\chi, s)$, in a similar way as the Anderson–Thakur special function interpolates Gauss–Thakur sums.

Remark. The Riemann zeta function (and Dirichlet L-functions in general) have a well-known functional equation. The formulation of an analogue functional equation for our Dirichlet-like series is still an open problem (see [Gos98][Subsection 8.1]).

Let's fix a primitive multiplicative character $\rho : {\mathbb{Z}/_{n\mathbb{Z}}}^\times \to \mathbb{C}^\times$ with $\rho(-1) = -1$; in this case, the following functional identity holds for all s:

$$\mathcal{L}(\bar{\rho}, s) = -2^s\pi^{s-1}n^{-\frac{s}{2}} i \sin\left((s+1)\frac{\pi}{2}\right)\Gamma(1-s)\mathcal{G}(\rho,\zeta)\mathcal{L}(\rho, 1-s), \tag{1.1}$$

where $\zeta : \mathbb{Z}/_{n\mathbb{Z}} \to \mathbb{C}^\times$ sends k to $\exp\left(\frac{k}{n}2\pi i\right)$. Through some analytic manipulation, it's possible to obtain the following explicit expression for the evaluation of the left hand side at $s = 0$:

$$\mathcal{L}(\bar{\rho}, 0) = -\frac{1}{n}\sum_{j\in\mathbb{Z}/n\mathbb{Z}^\times} \rho(j)^{-1}j. \tag{1.2}$$

If we evaluate the whole functional identity at $s = 0$ we get:

$$\frac{-i\pi}{n}\sum_{j\in\mathbb{Z}/n\mathbb{Z}^\times} \rho(j)^{-1}j = \mathcal{G}(\rho,\zeta)\mathcal{L}(\rho, 1). \tag{1.3}$$

Going back to the function field case, in [Pel12][Thm. 1], using two different methods (one involving modular forms, the other one using some log-algebraicity results of Anderson from [And94] and [And96]) Pellarin proved the following identity in $\mathbb{C}_\infty[[t]]$, connecting the evaluation of his L-function at $s = 1$ and the special function of Anderson–Thakur ω:

$$\frac{\tilde{\pi}}{t-\theta} = \omega L(1). \tag{1.4}$$

Let's fix the morphism $\xi : \mathbb{F}_q[t] \to \nu_{\mathfrak{p}}$ sending $a \in \mathbb{F}_q[t]$ to $\exp_C\left(a\tilde{\pi}\frac{\mathfrak{p}'}{\mathfrak{p}}\right)$. By Anglès and Pellarin's theorem [AP14][Thm. 2.9], if we evaluate the previous identity at an $\mathbb{F}_q$-linear homomorphism $\chi : \mathbb{F}_q[t] \to \overline{\mathbb{F}_q}$ sending t to some n-th root of unity, we deduce:

$$\frac{\tilde{\pi}}{\theta - \theta^{n+1}}\sum_{j=1}^{n} \chi(t)^{-j}\theta^j = \mathfrak{g}(\chi,\xi)L(\chi, 1), \tag{1.5}$$

which is remarkably similar to equation (1.3). In practice, Pellarin's identity (1.4) interpolates the equation (1.5) across all suitable characters for the fixed exponent $s = 1$, instead of interpolating across all suitable exponents for a fixed character χ, like what happens in the classical setting.

1.3 Drinfeld A-modules of rank 1

Let $X/_{\mathbb{F}_q}$ be a smooth, projective, geometrically irreducible curve of genus $g(X)$ with a closed point $\infty \in X$ of degree e. Let's denote by K the field of rational functions, and by A the ring $H^0(X \setminus \{\infty\}, \mathcal{O}_X)$ of rational functions with only poles at ∞; let's also denote by K_∞ the completion of K at ∞, and by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞.

We fix a rational function t with a pole of multiplicity 1 at ∞, so that $K_\infty \cong \mathbb{F}_{q^e}((t^{-1}))$; for all nonzero elements $c \in K_\infty^\times$, we denote by $\operatorname{sgn}(c) \in \mathbb{F}_{q^e}^\times$ its leading coefficient as an element of $\mathbb{F}_{q^e}((t^{-1}))$, by $\deg(c)$ its degree in the variable t multiplied by e, and by $\|c\| := q^{\deg(c)}$.

Let's denote by $\mathbb{C}_\infty[\tau]$ the noncommutative polynomial ring generated by τ with the relations $\tau c = c^q \tau$ for all $c \in \mathbb{C}_\infty$.

Definition. A *Drinfeld A-module of generic characteristic* is an $\mathbb{F}_q$-algebra homomorphism $\phi : A \to \mathbb{C}_\infty[\tau]$ such that for all $a \in A$ the constant term of ϕ_a is a.

Given a Drinfeld A-module ϕ of generic characteristic, there is a unique nonnegative integer r such that, for all $a \in A$, $r \cdot \deg(a) = \deg_\tau(\phi_a)$ (see [Gos98][Lemma 4.5.1, Prop. 4.5.3]). This constant is called the *rank* of ϕ.

In this thesis, when we write "Drinfeld module" we mean "Drinfeld A-module of generic characteristic and positive rank".

Definition. Two Drinfeld modules ϕ, ϕ' are said to be *isogenous* if there is an nonzero element $c \in \mathbb{C}_\infty[\tau]$ such that, for all $a \in A$, $\phi'_a \circ c = c \circ \phi_a$. If $c \in \mathbb{C}_\infty$, ϕ and ϕ' are said to be isomorphic.

If the leading term of ϕ_a is $\operatorname{sgn}(a)$ for all nonzero $a \in A$, ϕ is said to be *normalized*.

The Drinfeld module ϕ is said to be defined over the subring $R \subseteq \mathbb{C}_\infty$ if the coefficients of ϕ_a belong to R for all $a \in A$.

Remark. For all Drinfeld modules ϕ there is a normalized Drinfeld module ϕ' isomorphic to ϕ (see [Gos98][Thm. 7.2.15]).

Remark. If a Drinfeld module ϕ is defined over a ring $A \subseteq R \subseteq \mathbb{C}_\infty$, we can think of ϕ as a functor from R-algebras to A-modules sending an R-algebra S to the $\mathbb{F}_q$-vector space $\phi(S) := S$ endowed with the A-module structure induced by ϕ: for all $a \in A$, for all $s \in \phi(S)$, $a \cdot s := \phi_a(s)$.

Under this interpretation, if $A = \mathbb{F}_q[\theta]$ with the canonical sign, the Carlitz module is a normalized Drinfeld $\mathbb{F}_q[\theta]$-module of rank 1, i.e. an $\mathbb{F}_q$-algebra homomorphism $C : \mathbb{F}_q[\theta] \to \mathbb{C}_\infty[\tau]$ sending θ to $C_\theta := \theta + \tau$ (where we identify $\mathbb{F}_q[\theta]$ and $\mathbb{F}_q[t]$ in the original definition of the Carlitz module).

Similarly to the Carlitz module, given a Drinfeld module ϕ there is a surjective map of A-modules $\exp_\phi : \mathbb{C}_\infty \to \phi(\mathbb{C}_\infty)$; moreover, this map can be expressed as a formal series in $\mathbb{C}_\infty[[x]]$ which converges everywhere, and its kernel $\Lambda_\phi \subseteq \mathbb{C}_\infty$, called *period lattice*, is a discrete projective sub-A-module of the same rank as ϕ.

Moreover, for all discrete projective sub-A-modules $\Lambda \subseteq \mathbb{C}_\infty$ of rank r, there is a unique Drinfeld A-module of rank r of which Λ is the period lattice, and isomorphic Drinfeld modules correspond to isomorphic lattices (see [Gos98][Chapter 4]).

1.3.1 Hilbert class field

In the classical setting, the Hilbert class field of a given number field is its maximal abelian extension which is unramified at all places; this extension turns out to be finite, and its Galois group is naturally isomorphic to the ideal class group of the base field.

For the function field K, this definition is less useful, since there are infinite unramified abelian extensions, such as $\overline{\mathbb{F}_q}K$. In [Hay79], Hayes gave an alternative definition using Drinfeld A-modules of rank 1.

Definition. Let ϕ be a normalized Drinfeld A-module of rank 1. The *Hilbert class field* of A is the smallest field $H \subseteq \mathbb{C}_\infty$ such that ϕ is defined over H.

It turns out that the Hilbert class field does not depend on the choice of ϕ. Precisely, the following theorem holds.

Theorem ([Hay79][Prop. 8.4, Thm. 8.10]). *If H is the Hilbert class field of A, the extension ${}^{H}\!/_{K}$ is finite, unramified, Galois, and abelian, with field of constants $\mathbb{F}_{q^e}$, and its Galois group is naturally isomorphic to the ideal class group $Cl(A)$.*

Remark. In particular, if A has class number greater than 1, there are no Drinfeld modules of rank 1 defined over A.

Since there is a correspondence between Drinfeld A-modules and finitely generated sub-A-modules of $\mathbb{C}_\infty$, the set of Drinfeld modules of rank 1 up to isomorphism is in bijection with the ideal class group $Cl(A)$. On the other hand, since $Cl(A) \cong \mathrm{Gal}\left({}^{H}\!/_{K}\right)$ acts coefficient-wise on $H[\tau]$, it also acts on the set of Drinfeld modules defined over H.

In [Hay79], Hayes proved the following result, linking these two observations.

Theorem ([Hay79][Thm. 8.5]). *Let ϕ be a normalized Drinfeld A-module of rank 1. The set $\{\phi^\sigma\}_{\sigma \in \mathrm{Gal}\left({}^{H}\!/_{K}\right)}$ is a complete set of representatives for the Drinfeld modules of rank 1 up to isomorphism. Moreover, they are all isogenous to one another.*

1.3.2 Shtuka functions

Let's denote by $X_{\mathbb{C}_\infty}$ the base-changed curve $X \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(\mathbb{C}_\infty)$, and by $A_{\mathbb{C}_\infty} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$ the ring of rational functions with poles only above ∞. From now on, let's assume that $\infty \in X$ is $\mathbb{F}_q$-rational, so we can extend the sign $\mathrm{sgn} : K^\times \to \mathbb{F}_q^\times$ to a map from the nonzero rational functions on $X_{\mathbb{C}_\infty}$ to $\mathbb{C}_\infty^\times$.

For any divisor D on $X_{\mathbb{C}_\infty}$ let's denote by $D^{(1)}$ the pullback under the Frobenius automorphism $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty$; we adopt a similar notation for rational functions on $X_{\mathbb{C}_\infty}$ (see Subsection 3.3.1 for more details). Let's denote by $\Xi \in X(\mathbb{C}_\infty)$ the point corresponding to the natural map $\mathrm{Spec}(\mathbb{C}_\infty) \to \mathrm{Spec}(A) \subseteq X$.

Definition. A *Drinfeld divisor* V is a divisor of degree $g(X)$ such that $V^{(1)} - V + \Xi - \infty$ is a principal divisor. A rational function f on $X_{\mathbb{C}_\infty}$ with that divisor is called a *shtuka function*.

It can be proven that the infinite $\mathbb{F}_q$-vector space $H^0(V, (X \setminus \{\infty\})_{\mathbb{C}_\infty})$ is spanned by the finite products:

$$\left\{ e_d := \prod_{i=0}^{d-1} f^{(i)} \right\}_{d \geq 0}.$$

Since $1 \otimes a \in H^0(V, (X \setminus \{\infty\})_{\mathbb{C}_\infty})$ for all $a \in A$, we can write $1 \otimes a = \sum_i (a_i \otimes 1) e_i$; the map $A \to \mathbb{C}_\infty[\tau]$ sending a to $\sum_i a_i \tau^i$ is a Drinfeld module of rank 1 (see e.g. [Gos98][Section 7.11]). This correspondence determines a bijection between shtuka functions and Drinfeld modules of rank 1.

Remark. This is actually a particular case of a more general correspondence between Drinfeld modules of arbitrary rank—up to equivalence—and certain torsion-free coherent $\mathcal{O}_{X_{\mathbb{C}_\infty}}$-modules (see [Gos98][Section 6.2]).

Remark. Since by Hayes work in [Hay79] a Drinfeld module of rank 1 is defined over H up to isomorphism, it turns out that the Drinfeld divisor V is H-rational, and the shtuka function, up to scalar multiple, is a rational function on X_H.

1.3.3 Special functions

In their paper [ANT17a], Anglès, Ngo Dac, and Tavares Ribeiro proposed a generalization of the special function of Anderson–Thakur to Drinfeld-A-modules of rank 1. Let $\mathbb{C}_\infty \overline{\otimes} A$ denote the completion of the tensor product $A_{\mathbb{C}_\infty}$ with respect to the topology induced by $\mathbb{C}_\infty$.

Definition. Given a Drinfeld module ϕ of rank 1, an element $\omega \in \mathbb{C}_\infty \overline{\otimes} A$ is called a *special function* if, for all $a \in A$, $(\phi_a \otimes 1)\omega = (1 \otimes a)\omega$.

Remark. Given an $\mathbb{F}_q$-basis $\{a_i\}_{i \geq 0}$ of A,

$$\mathbb{C}_\infty \overline{\otimes} A = \left\{ \sum_{i \geq 0} c_i \otimes a_i | \lim_i c_i = 0 \right\}.$$

The series $\sum_i c_i \otimes a_i$ is a special function if and only if $\sum_i \phi_b(c_i) \otimes a_i = \sum_i c_i \otimes b a_i$ for all $b \in A$.

Anglès, Ngo Dac, and Tavares Ribeiro also proved that the set of special functions relative to a Drinfeld module of rank 1 is a projective A-module of rank 1 ([ANT17a][Rmk. 3.10]). They originally conjectured this module to be free, until Gazda and Maurischat proved the following result (see [GM21][Thm. 3.11]).

Theorem (Gazda–Maurischat)**.** *Let ϕ be a Drinfeld module. Denote by Λ_ϕ the period lattice of ϕ and Ω the module of Kähler differentials of A. The A-module of special functions is isomorphic to $\Lambda_\phi \otimes_A \Omega^{-1}$.*

The Frobenius automorphism of $\mathbb{C}_\infty$ induces an A-linear automorphism of $\mathbb{C}_\infty \overline{\otimes} A$, which we denote by τ. In Subsection 1.1.2, we observed that the Anderson–Thakur special function $\omega \in \mathbb{C}_\infty \overline{\otimes} \mathbb{F}_q[t] \subseteq \mathbb{C}_\infty[[t]]$ is such that $\tau\omega = (t - \theta)\omega$, and this functional identity actually implies that $(C_a \otimes 1)\omega = (1 \otimes a)\omega$ for all $a \in \mathbb{F}_q[t]$.

Anglès, Ngo Dac and Tavares Ribeiro proved a similar result in the case of Drinfeld A-modules of rank 1: an element $\omega \in \mathbb{C}_\infty \overline{\otimes} A$ is a special function if and only if $\tau\omega = f\omega$.

If f belongs to $\mathbb{C}_\infty \overline{\otimes} A$ and is invertible—as is the case for the Carlitz module—it's not difficult to construct an invertible special function $\omega \in (\mathbb{C}_\infty \overline{\otimes} A)^\times$ as an infinite product in a similar fashion as the Anderson–Thakur special function defined in Subsection 1.1.2.

Gazda and Maurischat noticed in [GM21][Cor. 3.22] that, if there is an invertible special function, then the A-module of special functions is free of rank 1, and conjectured that the reverse implication is true; in other words, they asked how restrictive is the hypothesis that $f \in (\mathbb{C}_\infty \overline{\otimes} A)^\times$. In this paper we answer affirmatively to this conjecture.

Theorem (Thm. 2.3.7)**.** *Let ϕ be a Drinfeld module of rank 1 and suppose that the module of special functions is free of rank 1. Then, there is a special function which is invertible as an element of $\mathbb{C}_\infty \overline{\otimes} A$.*

Moreover, we also prove that it is possible to construct a special function as an infinite product like in the Definition of the Anderson–Thakur special function without assuming that f is in $(\mathbb{C}_\infty \overline{\otimes} A)^\times$.

Theorem (Thm. 4.2.6)**.** *Fix a normalized Drinfeld module ϕ of rank 1 with shtuka function f. There is $\alpha \in K_\infty^\times$ such that the following element of $\mathbb{C}_\infty \overline{\otimes} K$ is well defined (up to the choice of a $(q-1)$-th root of α):*

$$\omega := (\alpha \otimes 1)^{\frac{1}{q-1}} \prod_{i \geq 0} \left(\frac{\alpha \otimes 1}{f} \right)^{(i)}.$$

Moreover, there is a nonzero special function $\omega' \in \mathbb{C}_\infty \overline{\otimes} A$ and a constant $c \in K^\times$ such that $\omega = (1 \otimes c)\omega'$.

1.4 A generalization of Pellarin's identity to Drinfeld A-modules of rank 1

Given an ideal $I < A$, it's possible to define an object of $\mathbb{C}_\infty \overline{\otimes} A$ which generalizes Pellarin's L-value $L(1)$.

Definition. Given a nonzero ideal $I < A$, the *partial Pellarin zeta function* ζ_I is defined as follows:

$$\zeta_I \coloneqq \sum_{a \in I \setminus \{0\}} a^{-1} \otimes a \in \mathbb{C}_\infty \overline{\otimes} A.$$

The partial zeta ζ_I, like $L(1)$, is a rigid analytic function on the analytification of the affine curve $(X \setminus \{\infty\})_{\mathbb{C}_\infty}$ (see [CNP23]); in other words, it's possible to define an evaluation of ζ_I at any point $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$ (see Subsection 4.3.1).

The following question is the starting point for this thesis.

Question. Given a Drinfeld module of rank 1, is the product of ζ_A with a special function a rational function on $X_{\mathbb{C}_\infty}$?

Remark. By the remark of Anglès, Ngo Dac and Tavares Ribeiro [ANT17a][Rmk. 3.10], for any two nonzero special functions $\omega, \omega' \in \mathbb{C}_\infty \overline{\otimes} A$ there is some $c \in K^\times$ such that $\omega' = (1 \otimes c)\omega$. In particular, $\zeta_A \omega$ is rational if and only if $\zeta_A \omega'$ is.

The first partial answer to this question is due to Green and Papanikolas, who proved that the answer is affirmative when X is an elliptic curve ([GP18][Thm. 7.1]). Let's denote by H the Hilbert class field of K.

Theorem (Green–Papanikolas)**.** *Suppose $g(X) = 1$, and fix the unique normalized Drinfeld module ϕ with period lattice $\tilde{\pi} A \subseteq \mathbb{C}_\infty$ for some $\tilde{\pi} \in \mathbb{C}_\infty^\times$. Let h be the unique rational function on X_H with* $\operatorname{sgn}(h) = 1$ *and divisor $V + (-V)^{(1)} - \Xi - \infty$. The A-module of special functions is free and generated by:*

$$\frac{(\tilde{\pi} \otimes 1)h}{\zeta_A}.$$

Let's fix an ideal $J < A$ of degree d_J such that J^{-1} is isomorphic to the Kähler module of differentials Ω. In the following theorem, given a Drinfeld module ϕ, we call its associated Drinfeld divisor V, and we denote by V_* the unique effective divisor of degree g such that the divisor $V + V_*^{(1)} - J - \Xi - (2g - d_J - 1)\infty$ is principal (see Proposition 3.3.25 and Lemma 4.3.30).

Theorem (Theorem 4.3.32)**.** *Let ϕ be a normalized Drinfeld module of rank 1 with period lattice $\tilde{\pi}_I I$, where $\tilde{\pi}_I \in \mathbb{C}_\infty^\times$ and $I < A$ is a nonzero ideal. Let h be the unique rational function on X_H with* $\mathrm{sgn}(h) = 1$ *and divisor $V + V_*^{(1)} - J - \Xi - (2g - d_J - 1)\infty$ such that the A-module of special functions in $\mathbb{C}_\infty \overline{\otimes} A$ is:*

$$\frac{(\tilde{\pi}_I \otimes 1)h}{\zeta_I}(\mathbb{F}_q \otimes IJ).$$

Remark. The element $\tilde{\pi}_I \in \mathbb{C}_\infty$ is uniquely determined up to a factor in $\mathbb{F}_q^\times$. Moreover, the module only depends on the ideal classes of I and J.

Remark. It's worth noting that the techniques employed by Green and Papanikolas were tailored to the case $g(X) = 1$ – for example, they choose a Weierstrass model of a generic elliptic curve to carry out explicit computations.

To prove Theorem 4.3.32—apart from the computations carried out in Section 4.3, which are needed to make explicit the scalar factor $\tilde{\pi}_I \otimes 1$—we employ the purely theoretical results of Section 3.1 and Section 3.3.

Remark. The explicit constant $\tilde{\pi}_I \otimes 1$ is a motivating reason behind the results proven in Chapter 5, which hold for Drinfeld modules of arbitrary rank. In retrospect, we can also use those results as an alternative way to derive the constant $\tilde{\pi}_I \otimes 1$.

1.5 "Dual" special functions in Drinfeld A-modules of rank 1

1.5.1 Adjoint Drinfeld modules

Let's denote by $\mathbb{C}_\infty[\tau^{-1}]$ the noncommutative polynomial ring generated by τ^{-1} with the relations $\tau^{-1}c^q = c\tau^{-1}$ for all $c \in \mathbb{C}_\infty$. There is an anti-isomorphism of algebras $\cdot^* : \mathbb{C}_\infty[\tau] \to \mathbb{C}_\infty[\tau^{-1}]$ sending $\sum_i a_i \tau^i$ to $\sum_i \tau^{-i} a_i$.

Definition. Given a Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$, its *adjoint* is the ring homomorphism $\phi^* : A \to \mathbb{C}_\infty[\tau^{-1}]$ sending $a \in A$ to $(\phi_a)^*$.

In analogy with the definition of special functions, we can give the following definition.

Definition. Given a Drinfeld module ϕ of rank 1, an element $\zeta \in \mathbb{C}_\infty \overline{\otimes} A$ is called a *dual special function* if, for all $a \in A$, $(\phi_a^* \otimes 1)\zeta = (1 \otimes a)\zeta$.

1.5.2 Pellarin zetas as dual special functions

Given a normalized Drinfeld module ϕ of rank 1, recall the definitions of the divisors V and V_*. In Section 4.1 we introduce the *adjoint shtuka function* f_* associated to ϕ as the unique rational function on X_{K_∞} with divisor $V_* - V_*^{(1)} + \Xi - \infty$ and $\mathrm{sgn}(f_*) = 1$. In analogy with Theorem 4.2.6, we prove the following identity.

Theorem (Theorem 4.3.28)**.** *Let $\tilde{\pi}_I I$ be the period lattice of ϕ, where $I < A$ is a nonzero ideal and $\tilde{\pi}_I \in \mathbb{C}_\infty^\times$, and fix $a_I \in I$ an element of least degree. The following identity holds in $\mathbb{C}_\infty \overline{\otimes} K$:*

$$\zeta_I = -(a_I^{-1} \otimes a_I) \prod_{i \geq 0} \left((\tilde{\pi}_I a_I \otimes 1)^{1-q} f_*^{(1)} \right)^{(i)}.$$

From the previous formula it's easy to derive the following identity (Proposition 4.3.24):

$$\left((\tilde{\pi}_I^{-1} \otimes 1)\zeta_I\right)^{(-1)} = f_* \left((\tilde{\pi}_I^{-1} \otimes 1)\zeta_I\right).$$

Its similarity with the defining identity of a special function ω—i.e. $\omega^{(1)} = f\omega$—suggests this formula as an alternative definition of Pellarin zeta functions. Indeed we can use Proposition 4.3.24 to deduce the following theorem.

Theorem (Theorem 4.3.27)**.** *Let $\tilde{\pi}_I I$ be the period lattice of ϕ, where $I < A$ is a nonzero ideal and $\tilde{\pi}_I \in \mathbb{C}_\infty^\times$, and let $\phi^* : A \to \mathbb{C}_\infty[\tau^{-1}]$ denote the adjoint Drinfeld module. Then $(\tilde{\pi}_I^{-1} \otimes 1)\zeta_I$ is a dual special function.*

While the original definition of Pellarin zeta functions does not depend on the choice of Drinfeld module ϕ, the definition of dual special functions does; moreover, like with special functions, they are well defined even if we don't assume the rank of ϕ to be 1. This allows us to make a conceptual leap previously impossible and prompts a series of questions for a Drinfeld module ϕ of arbitrary rank, namely:

- Is there always an explicit expression of dual special functions as a series like in the case of rank 1?
- Does a generalization of Pellarin's identity hold?

The answer to both questions is yes, and they are the main focus of Chapter 5.

1.6 Drinfeld A-modules of arbitrary rank

Let's drop the assumption $\infty \in X(\mathbb{F}_q)$. The main problem when trying to generalize Pellarin's identity to a Drinfeld module ϕ of arbitrary rank is the absence of a "canonical" special function and a "canonical" dual special function to multiply. If ϕ has rank 1, as stated in the first Remark of Section 1.4, we can overcome this problem by using any nonzero special function, because they are all rational multiples of one another. On the other hand, when ϕ has arbitrary rank r, Gazda and Maurischat proved in [GM21][Thm. 3.11] that the A-module of special functions has rank r, invalidating this line of reasoning.

If ϕ is a normalized Drinfeld A-module of rank 1 with period lattice $\tilde{\pi}_I I \subseteq \mathbb{C}_\infty$ for some ideal $I < A$ and some constant $\tilde{\pi}_I$, the most likely candidate for a "canonical" dual special function is $\tilde{\pi}_I^{-1}\zeta_I = \sum_{a \in I \setminus \{0\}} (\tilde{\pi}_I a)^{-1} \otimes a$. This prompts the following claim.

Claim. If ϕ is an arbitrary Drinfeld A-module with period lattice Λ_ϕ, the correct "canonical" object we need to consider is $\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda \in \mathbb{C}_\infty \overline{\otimes} \Lambda_\phi$.

Up to a sign, this claim is fundamentally correct, and helps to develop the correct framework to express the generalization of Pellarin's identity to arbitrary Drinfeld modules: the first step consists in generalizing the definitions of special functions and dual special functions.

1.6.1 Anderson eigenvectors and dual Anderson eigenvectors

In light of the previous consideration, in this thesis we give the following definition.

Definition. Let ϕ be a Drinfeld A-module. For any A-module M, the set of *Anderson eigenvectors* and *dual Anderson eigenvectors* relative to M are defined respectively as:

$$\mathrm{Sf}_\phi(M) \coloneqq \{\omega \in \mathbb{C}_\infty \overline{\otimes} M | (1 \otimes a)\omega = (\phi_a \otimes 1)\omega \ \forall a \in A\}$$
$$\mathrm{Sf}_{\phi^*}(M) \coloneqq \{\zeta \in \mathbb{C}_\infty \overline{\otimes} M | (1 \otimes a)\zeta = (\phi_a^* \otimes 1)\zeta \ \forall a \in A\}$$

We denote by $\mathrm{Sf}_\phi : A - \mathbf{Mod} \to A - \mathbf{Mod}$ and $\mathrm{Sf}_{\phi^*} : A - \mathbf{Mod} \to A - \mathbf{Mod}$ the natural functors that extend the maps above.

Remark. The A-modules $\mathrm{Sf}_\phi(A)$ and $\mathrm{Sf}_{\phi^*}(A)$ are respectively the module of special functions and the module of dual special functions.

The main tool we use to study Anderson eigenvectors and dual Anderson eigenvectors is the following result, which allows us to reinterpret the modules $\mathbb{C}_\infty \overline{\otimes} M$ as function spaces.

Proposition (Prop. 2.1.14)**.** *Let M be a discrete A-module. The set $\mathbb{C}_\infty \overline{\otimes} M$ is naturally isomorphic to the set of continuous $\mathbb{F}_q$-linear maps from $\mathrm{Hom}_{\mathbb{F}_q}(M, \mathbb{F}_q)$, endowed with the compact-open topology, to $\mathbb{C}_\infty$.*

Remark. We have that $\mathrm{Hom}_{\mathbb{F}_q}(\Omega, \mathbb{F}_q) \cong {K_\infty}/{A}$ ([Poon96, Thm. 8]). As a consequence, the space $\mathrm{Hom}_{\mathbb{F}_q}(\mathrm{Hom}_A(\Lambda_\phi, \Omega), \mathbb{F}_q)$ is isomorphic to ${K_\infty \Lambda_\phi}/{\Lambda_\phi}$.

With this new language at our disposal, it now makes sense to ask if the object

$$\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda \in \mathbb{C}_\infty \overline{\otimes} \Lambda_\phi$$

is a dual Anderson eigenvector.

We prove the following theorem, which provides us with a "canonical" Anderson eigenvector and a "canonical" dual Anderson eigenvector.

Theorem (Thm. 2.2.9,Thm. 5.2.10)**.** *Let ϕ be a Drinfeld module with period lattice Λ_ϕ. The functors Sf_ϕ and Sf_{ϕ^*} are represented respectively by $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ and Λ_ϕ. Moreover:*

- *the universal object of Sf_ϕ,*

$$\omega_\phi \in \mathbb{C}_\infty \overline{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega) \cong \mathrm{Hom}_{\mathbb{F}_q}^{cont}\left({K_\infty \Lambda_\phi}/{\Lambda_\phi}, \mathbb{C}_\infty\right),$$

  corresponds to the exponential $\exp_\phi$;

- *the universal object of Sf_{ϕ^*} is*

$$\zeta_\phi \coloneqq - \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda \in \mathbb{C}_\infty \overline{\otimes} \Lambda_\phi.$$

1.6.2 A generalization of Pellarin's identity

An unexpected consequence of being able to work with the universal Anderson eigenvector $\omega_\phi \in \mathbb{C}_\infty \overline{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega)$ and the universal dual Anderson eigenvector $\zeta_\phi \in \mathbb{C}_\infty \overline{\otimes} \Lambda_\phi$ is that we have a natural candidate for a "product" between the two: we may simply take the image of the pair $(\zeta_\phi, \omega_\phi)$ under the natural $\mathbb{C}_\infty \overline{\otimes} A$-bilinear pairing

$$\_ \cdot \_ : \mathbb{C}_\infty \overline{\otimes} \Lambda_\phi \times \mathbb{C}_\infty \overline{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega) \to \mathbb{C}_\infty \overline{\otimes} \Omega \cong \mathrm{Hom}_{\mathbb{F}_q}^{cont}\left({K_\infty}/{A}, \mathbb{C}_\infty\right).$$

This allows us to state and prove the following generalization of Pellarin's identity.

Theorem (Thm. 5.4.2)**.** *Let ϕ be a Drinfeld A-module with period lattice Λ_ϕ, and denote by ω_ϕ and ζ_ϕ the universal objects of the functors Sf_ϕ and Sf_{ϕ^*}, respectively. For all integers k, the pairing $\zeta_\phi \cdot \left(\tau^k \omega_\phi\right)$ in $\mathbb{C}_\infty \overline{\otimes} \Omega$ is a rational differential form on $X_{\mathbb{C}_\infty}$.*

In contrast with Theorem 4.3.32, we are not able to describe the rational differential forms $\zeta_\phi \cdot \left(\tau^k \omega_\phi\right)$ in terms of their divisor. On the other hand, we are able to describe it as a function from ${K_\infty}/{A}$ to $\mathbb{C}_\infty$ in terms of the coefficients of the Drinfeld module ϕ.

Definition. Let $\Phi : K_\infty \to \mathbb{C}_\infty[[\tau]]$ be the unique extension of ϕ which is multiplicative and coefficient-wise continuous.

Let $\hat{\Phi} : K_\infty \to \mathbb{C}_\infty[[\tau]][\tau^{-1}]$ be the unique extension of ϕ^* which is multiplicative and coefficient-wise continuous.

Remark. If we denote by $\log_\phi \in \mathbb{C}_\infty[[\tau]]$ the formal inverse of $\exp_\phi$, Φ sends $c \in K_\infty$ to $\exp_\phi \circ c \circ \log_\phi$. For the existence of $\hat{\Phi}$, see e.g. Proposition 5.4.15.

Theorem (Thm. 5.4.17)**.** *The following identity holds in the space $\mathbb{C}_\infty[[\tau, \tau^{-1}]]$ of formal bilateral series for all $c \in K_\infty$:*

$$\sum_{k \in \mathbb{Z}} \left(\zeta_\phi \cdot (\tau^k \omega_\phi)\right)(c)\tau^k = (\Phi_c)^* - \hat{\Phi}_c.$$

We can use the previous theorem to derive the rational differential forms $\zeta_\phi \cdot (\tau^k \omega_\phi)$ for any given Drinfeld module. For example, we carry out this computation when:

- $A = \mathbb{F}_q[\theta]$, r is arbitrary, and $k = 0, \ldots, r-1$ (Proposition 5.5.1);
- when A comes from a hyperelliptic curve, $r = 1$, and $k = 0$ (Theorem 5.5.12).

In the latter case, we also write the shtuka function in terms of the coefficients of the Drinfeld module ϕ.

1.7 Anderson A-modules

An Anderson A-module $\underline{E} = (E, \phi)$ of dimension d (see [HJ20][Def. 2.5.2]) consists of:

- an $\mathbb{F}_q$-module scheme E over $\mathbb{C}_\infty$ isomorphic to $\mathbb{G}_a^d$;
- an action $\phi : A \to \mathrm{End}(E)$ such the induced action $\mathrm{Lie}(\phi)$ on the tangent space $\mathrm{Lie}(E) \cong \mathbb{C}_\infty^d$ sends $a \in A$ to $a \cdot \mathrm{Id}_d$ plus a nilpotent endomorphism.

In other words, $\underline{E}$ is a functor from $\mathbb{C}_\infty - \mathbf{Alg}$ to $A - \mathbf{Mod}$ such that, for any $\mathbb{C}_\infty$-algebra S, $E(S)$ is naturally isomorphic to S^d as an $\mathbb{F}_q$-vector space, and its A-module structure must satisfy some technical conditions (see also Subsection 2.2.1 for more details).

Remark. Drinfeld A-modules correspond to the Anderson A-modules of dimension 1.

As for the Carlitz module and the Drinfeld A-modules, there is an A-linear map

$$\exp_\phi : \mathrm{Lie}(E) \to E(\mathbb{C}_\infty)$$

whose kernel Λ_ϕ is a discrete sub-A-module of $\mathrm{Lie}(E)$. If we fix an isomorphism

$$\mathrm{Lie}(E) \cong E(\mathbb{C}_\infty) \cong \mathbb{C}_\infty^d,$$

$\exp_\phi$ can be expressed as an everywhere converging series in $(\mathbb{C}_\infty[[x_1, \ldots, x_d]])^d$. Moreover, if we denote by $\tau : \mathbb{C}_\infty^d \to \mathbb{C}_\infty^d$ the map sending a vector $(v_i)_i$ to $(v_i^q)_i$, the exponential can actually be expressed as an infinite series:

$$\exp_\phi = \sum_{i \geq 0} E_i \tau^i \in \mathbb{C}_\infty[[\tau]],$$

where the E_i's are d-by-d matrices and E_0 is the identity (see [Gos98][Section 5.9]).

1.7.1 Gauss–Thakur sums, special functions, and Anderson eigenvectors

In the paper [GM21], where Gazda and Maurischat proposed the following definitions to generalize the concepts of Gauss–Thakur sums and special functions to Anderson modules.

Definition ([GM21][Def. 3.1])**.** Let (E, ϕ) be an Anderson A-module. A special function is an element $\omega \in E(\mathbb{C}_\infty)\overline{\otimes}A$ such that, for all $a \in A$:

$$(\phi_a \otimes 1)\omega = (1 \otimes a)\omega.$$

Remark. The theorem [GM21][Thm. 3.11] actually holds in this generality.

Definition ([GM21][Def 4.1])**.** Fix a nonzero prime ideal $\mathfrak{p} < A$ and a multiplicative character $\chi : {A/\mathfrak{p}}^\times \to {A/\mathfrak{p}}^\times$. Denote by $E[\mathfrak{p}](\mathbb{C}_\infty) \subseteq E(\mathbb{C}_\infty)$ the $\mathfrak{p}$-torsion subset and fix an A-linear map $\xi : A/\mathfrak{p} \to E[\mathfrak{p}](\mathbb{C}_\infty)$. The (tensor) Gauss–Thakur sum relative to χ and ξ is defined as follows:

$$\mathfrak{g}(\chi, \xi) := \sum_{a \in A/\mathfrak{p}} \xi(a) \otimes \chi(a)^{-1} \in E(\mathbb{C}_\infty) \otimes_{\mathbb{F}_q} A/\mathfrak{p}.$$

The space of Gauss–Thakur sums is defined as:

$$G(\phi, \chi) := \left\{ \mathfrak{g} \in E(\mathbb{C}_\infty) \otimes_{\mathbb{F}_q} A/\mathfrak{p} | (\phi_a \otimes 1)\mathfrak{g} = (1 \otimes a)\mathfrak{g} \right\}$$

They also prove that, if χ cannot be lifted to a morphism of $\mathbb{F}_q$-algebras $\chi : A/\mathfrak{p} \to A/\mathfrak{p}$, $\mathfrak{g}(\chi, \xi) = 0$ for any choice of ξ ([GM21][Prop. 4.8]) and that $G(\phi, \chi)$ is spanned by $\{\mathfrak{g}(\chi, \xi)\}_\xi$ as an $A/\mathfrak{p}$-vector space.

Like for Drinfeld modules, we define the Anderson eigenvectors for any Anderson module.

Definition. Let (E, ϕ) be an Anderson A-module. For any A-module M, the set of *Anderson eigenvectors* is defined as:

$$\mathrm{Sf}_\phi(M) := \{\omega \in E(\mathbb{C}_\infty)\overline{\otimes}M | (1 \otimes a)\omega = (\phi_a \otimes 1)\omega \ \forall a \in A\}.$$

We denote by $\mathrm{Sf}_\phi : A - \mathbf{Mod} \to A - \mathbf{Mod}$ the natural functor that extend this map.

Remark. Fix a nonzero prime ideal $\mathfrak{p} < A$. If $\chi : A/\mathfrak{p} \to A/\mathfrak{p}$ is a morphism of $\mathbb{F}_q$-algebras, the space of Gauss–Thakur sums $G(\phi, \chi)$ coincides with $\mathrm{Sf}_\phi\left(A/\mathfrak{p}\right)$, where the A-module structure of $A/\mathfrak{p}$ is the one induced by χ. Moreover, the module of special functions coincides with $\mathrm{Sf}_\phi(A)$.

We prove a representability result in this generality.

Theorem (Thm. 2.2.9, Thm. 2.2.10)**.** *Let (E, ϕ) be an Anderson A-module. If either (E, ϕ) is uniformizable or we restrict the functor Sf_ϕ to the category of torsionless A-modules, Sf_ϕ is represented by the A-module $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$.*

1.7.2 Dual Anderson eigenvectors for arbitrary Anderson modules

Let (E, ϕ) be an Anderson A-module. Similarly to the case of Drinfeld modules, the A-module structure determined by ϕ on $E(\mathbb{C}_\infty)$ induces an adjoint action ϕ^* of A on

$$E(\mathbb{C}_\infty)^\vee := \mathrm{Hom}_{\mathbb{C}_\infty}(E(\mathbb{C}_\infty), \mathbb{C}_\infty).$$

This allows us to define the dual Anderson eigenvectors as follows.

Definition. Let (E, ϕ) be an Anderson A-module. For any A-module M, the set of *dual Anderson eigenvectors* is defined as:

$$\mathrm{Sf}_{\phi^*}(M) := \{\omega \in E(\mathbb{C}_\infty)^\vee \overline{\otimes} M | (1 \otimes a)\zeta = (\phi_a^* \otimes 1)\zeta \ \forall a \in A\}.$$

We denote by $\mathrm{Sf}_{\phi^*} : A - \mathbf{Mod} \to A - \mathbf{Mod}$ the natural functor that extend this map.

In the study of dual Anderson eigenvectors for Drinfeld modules in Chapter 5, we expand on a previous work by Poonen ([Poon96]): among other results, he studies the adjoint of the exponential function $\exp_\phi^* : \mathbb{C}_\infty \to \mathbb{C}_\infty$ and proves that its kernel is isomorphic to the Pontryagin dual of the period lattice Λ_ϕ ([Poon96][Thm. 10]).

Unfortunately, his techniques fail in higher dimensions, hence they can't be used to prove that the functor Sf_{ϕ^*} is representable, or that a generalization of Pellarin's identity holds, for arbitrary Anderson modules.

The last chapter of this thesis is dedicated to explore conjectural generalizations of the main theorems in the generality of Anderson A-modules, and links with the theory developed by Hartl and Juschka in [HJ20].

For simplicity, let's fix an isomorphism $\mathrm{Lie}(E) \cong E(\mathbb{C}_\infty) \cong \mathbb{C}_\infty^d$, so that we can write

$$\exp_\phi = \sum_{k \geq 0} E_k \tau^k \in \mathbb{C}_\infty[[\tau]],$$

where the E_i's are d-by-d matrices and E_0 is the identity. Since E_0 is invertible, we can also define the logarithm as the formal inverse of $\exp_\phi$:

$$\log_\phi = \sum_{k \geq 0} L_k \tau^k \in \mathbb{C}_\infty[[\tau]]$$

If we assume that Sf_{ϕ^*} is representable by Λ_ϕ, one of the conjectures relates the coefficients of its universal object $\zeta_\phi \in E(\mathbb{C}_\infty)^\vee \overline{\otimes} \Lambda_\phi$ with the coefficients of $\log_\phi$.

Conjecture (Conjecture 6.3.9). *Let $(\lambda_i)_i$ be an $\mathbb{F}_q$-linear basis of Λ_ϕ. If*

$$\zeta_\phi = \sum_i z_i \otimes \lambda_i \in E(\mathbb{C}_\infty)^\vee \overline{\otimes} \Lambda_\phi$$

is the universal dual Anderson eigenvector, for all $k \in \mathbb{Z}$ and all $v \in E(\mathbb{C}_\infty)$ the series

$$\sum_i \lambda_i \cdot (\tau^k \circ z_i(v))$$

converges in $\mathrm{Lie}(E)$ *and is equal to $L_k(v)$.*

If we consider a Drinfeld module $(\mathbb{G}_a, \phi)$, we know that the universal dual Anderson eigenvector exists and is equal to $-\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda$. The previous conjecture is meant to be a generalization of the well-known fact that, for all positive integers k, the k-th coefficient of the logarithm $\log_\phi \in \mathbb{C}_\infty[[\tau]]$ is equal to $-\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{1-q^k}$ (see [Gek88][Eqq. 2.8,2.9]).

When (E, ϕ) is the tensor power of the Carlitz module (see [Gos98][Section 5.8]), if we assume that Sf_{ϕ^*} is representable by Λ_ϕ, we manage to describe the universal dual Anderson eigenvector ζ_ϕ (Proposition 6.4.3) and use an explicit formula for the coefficients of the logarithm due to Papanikolas ([Pap15]) to prove that Conjecture 6.3.9 holds (Proposition 6.4.5).

Acknowledgements

I am grateful to Prof. Nathan Green, Dr. Quentin Gazda, Prof. Andreas Maurischat, and Dr. Oğuz Gezmiş for the fruitful conversations on the topics of this thesis.
I am thankful to Profs. Dinesh Thakur, Mihran Papikian, Gebhard Böckle, Tuan Ngo Dac, and Matt Papanikolas, for the interest they have shown in my work and for their continued willingness to support me.
Last but not least, this thesis would not have been possible without the aid and guidance of my advisor, Prof. Federico Pellarin.

Chapter 2

A generalization of special functions for Anderson A-modules

In this chapter, $X, A, K_\infty, \mathbb{C}_\infty$ are defined as in Section 1.3, and so are the degree map $\deg : A \to \mathbb{Z}$ and the norm $\|\cdot\| : \mathbb{C}_\infty \to \mathbb{R}_{>0}$.

As we explained in the introduction, the special function introduced by Anderson and Thakur in [AT90] has undergone successive generalizations: first to Drinfeld modules by Anglès, Ngo Dac, and Tavares Ribeiro ([ANT17a]), and then to Anderson modules by Gazda and Maurischat ([GM21]).

In this chapter, given an Anderson module $\underline{E} = (E, \phi)$, we introduce the functor of Anderson eigenvectors $\mathrm{Sf}_\phi : A - \mathbf{Mod} \to A - \mathbf{Mod}$ and prove that the module of special functions arises from the application of this functor to the object $A \in A - \mathbf{Mod}$.

In the main theorem of this chapter (Theorem 2.2.9) we prove that, if $\underline{E}$ is uniformizable, Sf_ϕ is representable, and we relate its universal object to the exponential map $\exp_\phi : \mathrm{Lie}(E) \to E(\mathbb{C}_\infty)$.

We apply this result in Section 2.3 to answer affirmatively the following conjecture of Gazda and Maurischat (see Theorem 2.3.7).

Conjecture ([GM21][Question]). *Let $(\mathbb{G}_a, \phi)$ be a Drinfeld module. If the A-module of special functions is free of rank 1, there is an invertible special function in $\mathbb{C}_\infty \overline{\otimes} A$.*

2.1 Pontryagin duality of A-modules

In this section, we give a formal definition of the completed tensor product $\mathcal{C}\overline{\otimes}M$, where $\mathcal{C}$ is a certain type of complete topological $\mathbb{F}_q$-vector space and M is a discrete $\mathbb{F}_q$-vector space (Definition 2.1.12).

In the main proposition (Proposition 2.1.14) we prove that $\mathcal{C}\overline{\otimes}M$ is naturally isomorphic to the space of continuous $\mathbb{F}_q$-linear maps from the Pontryagin dual of M to $\mathcal{C}$. This result is instrumental in understanding the generalization of the module of special functions presented in Section 2.2

2.1.1 Basic statements about Pontryagin duality

Throughout this thesis, compact and locally compact topological spaces are always assumed to be Hausdorff.

Definition 2.1.1 (Pontryagin duality)**.** Denote by $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ the complex unit circle. For any commutative ring with unity B, the *Pontryagin duality* is a contravariant functor from the category of topological B-modules to itself, sending a module M to the set of continuous group homomorphism

$\hat{M} := \mathrm{Hom}_{\mathbb{Z}}^{cont}(M, \mathbb{S}^1)$, endowed with the compact open topology and with the natural B-module structure.

Let's state the first main properties of Pontryagin duality for B-modules. They follow immediately from the analogous properties for $\mathbb{Z}$-modules, that are proven in Pontryagin's original article [Pon34].

Proposition 2.1.2. *For any ring B and any topological B-module M, consider the group homomorphism $i_M : M \to \hat{\hat{M}}$ sending $m \in M$ to $(f \mapsto f(m))$. The map i_M is a continuous B-linear homomorphism. If M is locally compact, $\hat{M}$ is locally compact, and i_M is an isomorphism. Moreover, if M is compact (resp. discrete) $\hat{M}$ is discrete (resp. compact).*

Remark 2.1.3. If M is an A-module, since M is also an $\mathbb{F}_q$-vector space, we have the following natural isomorphisms of topological A-modules:

$$\hat{M} := \mathrm{Hom}_{\mathbb{Z}}^{cont}(M, \mathbb{S}^1) \cong \mathrm{Hom}_{\mathbb{F}_q}^{cont}(M, \mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_q, \mathbb{S}^1)) = \mathrm{Hom}_{\mathbb{F}_q}^{cont}(M, \hat{\mathbb{F}}_q).$$

We fix an isomorphism $\mathbb{F}_q \cong \hat{\mathbb{F}}_q$ so that from now on, to ease notation, we can write $\hat{M} = \mathrm{Hom}_{\mathbb{F}_q}^{cont}(M, \mathbb{F}_q)$ for any $\mathbb{F}_q$-vector space M. Let's fix some additional notation.

Definition 2.1.4. Let M and N be topological $\mathbb{F}_q$-vector spaces with N locally compact. We define the topological tensor product of M and N the space $\mathrm{Hom}_{\mathbb{F}_q}^{cont}(\hat{N}, M)$ of continuous $\mathbb{F}_q$-linear homomorphisms from $\hat{N}$ to M, and we denote it by $M \hat{\otimes} N$.

Remark 2.1.5. The topological tensor product can be endowed with the compact open topology, but we only need to use the definition of the underlying set.

Lemma 2.1.6. *For any pair of locally compact A-modules M, N, there is a natural isomorphism of $A \otimes A$-modules between $M \hat{\otimes} N$ and $N \hat{\otimes} M$.*

Proof. By Proposition 2.1.2, the Pontryagin duality induces an antiequivalence of the category of locally compact $\mathbb{F}_q$-vector spaces with itself, hence we have the following natural bijections:

$$\mathrm{Hom}_{\mathbb{F}_q}^{cont}(\hat{N}, M) \cong \mathrm{Hom}_{\mathbb{F}_q}^{cont}(\hat{M}, \hat{\hat{N}}) \cong \mathrm{Hom}_{\mathbb{F}_q}^{cont}(\hat{M}, N);$$

the $A \otimes A$-linearity is a simple check. □

Remark 2.1.7. For any set I, the Pontryagin dual of $\mathbb{F}_q^{\oplus I}$ can be identified with $\mathbb{F}_q^I$. In particular, for any discrete $\mathbb{F}_q$-vector space M, an isomorphism $\mathbb{F}_q^{\oplus I} \cong M$, i.e. an $\mathbb{F}_q$-basis $(m_i)_{i \in I}$, induces an isomorphism of topological vector spaces between $\mathbb{F}_q^I = \widehat{\mathbb{F}_q^{\oplus I}}$ and $\hat{M}$.

We introduce some other useful terminology.

Definition 2.1.8. If M is a discrete $\mathbb{F}_q$-vector space with basis $(m_i)_{i \in I}$, for all $i \in I$ we denote by m_i^* the image of $(\delta_{i,j})_{j \in I} \in \mathbb{F}_q^I$ via the isomorphism with $\hat{M}$, so that for all $j \in I$ $m_i^*(m_j) = \delta_{i,j}$. We call $(m_i^*)_{i \in I}$ the *dual basis* of $\hat{M}$ relative to $(m_i)_{i \in I}$.

Remark 2.1.9. In the previous definition, an element $f \in \hat{M}$ corresponds to $(f(m_i))_i \in \mathbb{F}_q^I$. It's immediate to check that, for all $m \in M$, $f(m) = \sum_{i \in I} f(m_i) m_i^*(m)$, which is actually a finite sum, hence we are justified in the use the following formal notation: $f = \sum_{i \in I} f(m_i) m_i^*$. The existence and uniqueness of this expression for all $f \in \hat{M}$ explains the terminology "dual basis" for $(m_i^*)_i$.

2.1.2 Application to A-modules

Denote by Ω the module of Kähler differentials of A, which is a projective A-module of rank 1.

The following (see [Poon96, Thm. 8]) is a fundamental result about the Pontryagin duality of A-modules.

Theorem 2.1.10 (Poonen)**.** *There is a natural perfect pairing between $\Omega \otimes_A K_\infty$ and K_∞, which restricts to a perfect pairing between the discrete A-module Ω and the compact A-module ${K_\infty}/{A}$. In other words, $\widehat{\Omega \otimes_A K_\infty} \cong K_\infty$ and $\hat{\Omega} \cong {K_\infty}/{A}$.*

Remark 2.1.11. For any discrete projective A-module Λ of finite rank r, we have the following natural isomorphisms of topological A-modules, where $\Lambda^* := \mathrm{Hom}_A(\Lambda, A)$:

$$\widehat{\Lambda^* \otimes_A \Omega} = \mathrm{Hom}_{\mathbb{F}_q}(\Lambda^* \otimes_A \Omega, \mathbb{F}_q) \cong \mathrm{Hom}_A(\Lambda^*, \mathrm{Hom}_{\mathbb{F}_q}(\Omega, \mathbb{F}_q)) \cong \Lambda \otimes_A (K_\infty/A).$$

Retracing the isomorphisms, it's easy to check that the pairing $(\Lambda^* \otimes_A \Omega) \otimes_{\mathbb{F}_q} \left(\Lambda \otimes_A {K_\infty}/{A}\right) \to \mathbb{F}_q$ sends the element $(\lambda^* \otimes \omega) \otimes (\lambda \otimes b)$ to the image of $\lambda^*(\lambda) b \otimes \omega$ under the pairing ${K_\infty}/{A} \otimes_{\mathbb{F}_q} \Omega \to \mathbb{F}_q$.

We now show that in some cases the topological tensor product of two spaces is naturally isomorphic to a completion of their tensor product. This makes our notation agree with the usual notation $\mathbb{C}_\infty \hat{\otimes} A$ employed for the Tate algebra in works like [GM21], and others.

Definition 2.1.12. Let C be a topological vector spaces which is the projective limit of a diagram of discrete $\mathbb{F}_q$-vector spaces $\{C_i\}_{i \in I}$: we call such a space a *prodiscrete* $\mathbb{F}_q$-vector space. The collection $\mathcal{U} := \{\ker(C \to C_i)\}_{i \in I}$ is a neighborhood filter of 0 composed of clopen subspaces of C; we call it its *associated filter*.

For any discrete F_q-vector space M and any prodiscrete $\mathbb{F}_q$-vector space C, we denote by $C \overline{\otimes} M$ the completion of $C \otimes M$ with respect to the neighborhood filter of 0 given by $\{U \otimes M\}_{U \in \mathcal{U}}$.

Remark 2.1.13. Given a radius $r \in \mathbb{R}_{>0}$, the open ball

$$B_r = \{c \in \mathbb{C}_\infty \mid \|c\| < r\} \subseteq \mathbb{C}_\infty$$

is an $\mathbb{F}_q$-vector space, because the norm on $\mathbb{C}_\infty$ is nonarchimedean. Since $\mathbb{C}_\infty$ is complete, $\mathbb{C}_\infty$ is a prodiscrete $\mathbb{F}_q$-vector space, with associated filter $\{B_r\}_{r \in \mathbb{R}_{>0}}$.

Proposition 2.1.14. *Let C be a prodiscrete $\mathbb{F}_q$-vector space and let M be a discrete $\mathbb{F}_q$-vector space. There is a natural $\mathbb{F}_q$-linear bijection $\Phi : C \overline{\otimes} M \to C \hat{\otimes} M$. If we fix an $\mathbb{F}_q$-basis $(m_i)_{i \in I}$ of M with corresponding dual basis $(m_i^*)_{i \in I}$ of $\hat{M}$, for any function $f \in C \hat{\otimes} M = \mathrm{Hom}_{\mathbb{F}_q}^{cont}(\hat{M}, C)$ we have $\Phi^{-1}(f) = \sum_i f(m_i^*) \otimes m_i$.*

Moreover, if C and M are A-modules, Φ is $A \otimes A$-linear.

Proof. Fix an $\mathbb{F}_q$-basis $(m_i)_{i \in I}$ of M and let $\mathcal{U}$ be an associated filter of C. Any $x \in C \overline{\otimes} M$ can be expressed in a unique way as $\sum_{i \in I} x_i \otimes m_i$, where $x_i \in C$ for all $i \in I$, and for all $U \in \mathcal{U}$ the set $I_U := \{i \in I | x_i \notin U\}$ is finite. We define $\Phi(x) : \hat{M} \to C$ as follows:

$$\forall f \in \hat{M},\ \Phi(x)(f) := \lim_{\substack{J \subseteq I \\ \#J < \infty}} \sum_{i \in J} f(m_i) x_i.$$

Since C is complete with respect to the neighborhood filter $\mathcal{U}$, and for all $U \in \mathcal{U}$ the set $\{i \in I | f(m_i) x_i \notin U\} \subseteq I_U$ is finite, the map $\Phi(x)$ is well defined.

For all $U \in \mathcal{U}$, the set $\{f \in \hat{M} | f(m_i) = 0 \, \forall i \in I_U\}$ is a neighborhood of 0 in $\hat{M}$, and is contained in $\Phi(x)^{-1}(U)$, hence $\Phi(x)$ is continuous. Since $\Phi(x)$ is also obviously $\mathbb{F}_q$-linear, $\Phi(x) \in C\hat{\otimes}M$ for all $x \in C\overline{\otimes}M$.

The map Φ is clearly $\mathbb{F}_q$-linear. If C and M are A-modules, Φ is also $A \otimes A$-linear, since for any $x = \sum_i x_i \otimes m_i \in \mathbb{C}_\infty \overline{\otimes} M$ and any $a, b \in A$ we have the following identity for all $f \in \hat{M}$:

$$\Phi((a \otimes b)x)(f) = \Phi\left(\sum_i ax_i \otimes bm_i\right)(f) = \sum_i f(bm_i)(ax_i) = a\left(\sum_i (b \cdot f)(m_i)x_i\right)$$
$$= a\left(\Phi(x)(b \cdot f)\right) = \left((a \otimes b) \cdot \Phi(x)\right)(f).$$

We just need to prove bijectivity. On one hand, if $\Phi(x) \equiv 0$, we have $0 = \Phi(x)(m_i^*) = x_i$ for all $i \in I$, hence $x = 0$. On the other hand, if $g : \hat{M} \to C$ is a continuous function, for all $U \in \mathcal{U}$ the set $\{i \in I | g(m_i^*) \notin U\}$ is finite because $\hat{M}$ is compact, hence $y := \sum_i g(m_i^*) \otimes m_i$ is an element of $C\overline{\otimes}M$; since $\Phi(y)(m_i^*) = g(m_i^*)$ for all $i \in I$, $\Phi(y) = g$. □

2.2 Universal Anderson eigenvector

In this section, we define the functor of *Anderson eigenvectors* relative to an Anderson A-module (E, ϕ), which generalizes the concept of special functions and Gauss-Thakur sums (see Definition 2.2.7), and prove that under some conditions it is representable (see Theorem 2.2.9 and Theorem 2.2.10). As a corollary, we get a variant of the result [GM21][Thm. 3.11], in which Gazda and Maurischat described the module of special functions for any Anderson A-module (E, ϕ).

2.2.1 Anderson A-modules

Definition 2.2.1. Given an $\mathbb{F}_q$-algebra R, an *R-module scheme over* $\mathbb{C}_\infty$ is a group scheme over $\mathbb{C}_\infty$ endowed with a compatible action of R.

If G is a group scheme over $\mathbb{C}_\infty$, we denote by $\mathrm{Lie}(G)$ its tangent space at the identity, which has a natural structure of $\mathbb{C}_\infty$-vector space. This association can be extended to a functor from the category of group schemes over $\mathbb{C}_\infty$ to that of $\mathbb{C}_\infty$-vector spaces, and given $f : G \to G'$ a morphism in the first category, we denote the induced morphism $\mathrm{Lie}(G) \to \mathrm{Lie}(G')$ as $\mathrm{Lie}(f)$.

Let's recall the definition of Anderson A-modules (see [HJ20][Def. 2.5.2]).

Definition 2.2.2. An *Anderson A-module* $\underline{E} = (E, \phi)$ over $\mathbb{C}_\infty$ of dimension d consists of an A-module scheme E over $\mathbb{C}_\infty$ with the following properties:

- as an $\mathbb{F}_q$-module scheme over $\mathbb{C}_\infty$, E is isomorphic to $\mathbb{G}_{a,\mathbb{C}_\infty}^d$;
- the action ϕ of A on E is such that $\mathrm{Lie}(\phi_a) - a : \mathrm{Lie}(E) \to \mathrm{Lie}(E)$ is nilpotent for all $a \in A$.

Fix an Anderson A-module (E, ϕ). The following proposition sums up various basic results about (E, ϕ) (see [Gos98][Thm. 5.9.6] and [Gos98][Lemma 5.1.9] for proofs).

Proposition 2.2.3. *There is a nonzero $\mathbb{F}_q$-linear function $\exp_\phi : \mathrm{Lie}(E) \to E(\mathbb{C}_\infty)$, called exponential, such that $\exp_\phi \circ \mathrm{Lie}(\phi_a) = \phi_a \circ \exp_\phi$ for all $a \in A$; its kernel $\Lambda_\phi \subseteq \mathrm{Lie}(E)$ is an A-module of finite rank (with respect to the A-module structure induced by $\mathrm{Lie}(\phi)$ on $\mathrm{Lie}(E)$).*

Remark 2.2.4. Since E and $\mathbb{G}_{a,\mathbb{C}_\infty}^d$ are isomorphic group schemes over $\mathbb{C}_\infty$, and the group of automorphisms of $\mathbb{G}_{a,\mathbb{C}_\infty}^d$ as a group scheme over $\mathbb{C}_\infty$ is $\mathrm{GL}_{n,\mathbb{C}_\infty}$, we can identify the set $E(\mathbb{C}_\infty)$ with $\mathbb{G}_{a,\mathbb{C}_\infty}^d(\mathbb{C}_\infty) = \mathbb{C}_\infty^d$ up to an element of $\mathrm{GL}_{n,\mathbb{C}_\infty}(\mathbb{C}_\infty)$. In particular, $E(\mathbb{C}_\infty)$ has a natural structure of finite $\mathbb{C}_\infty$-vector space, hence, since $\mathbb{C}_\infty$ is a complete normed field, a natural structure of complete topological vector space over $\mathbb{C}_\infty$.

Similarly, $\mathrm{Lie}(E)$ also has a natural structure of complete topological vector space over $\mathbb{C}_\infty$; moreover, since $\exp_\phi$ is a local homeomorphism, we get that $\Lambda_\phi \subseteq \mathrm{Lie}(E)$ is a discrete subset (see [Gos98][5.9.12]). In light of this remark, and since for all $a \in A$ $\exp_\phi \circ \mathrm{Lie}(\phi_a) = \phi_a \circ \exp_\phi$, $\exp_\phi$ is a morphism of topological A-modules.

Definition 2.2.5. Let $\underline{E} = (E, \phi)$ be an Anderson A-module. The discrete A-module $\Lambda_\phi \subseteq \mathrm{Lie}(E)$ is called *period lattice*. If $\exp_\phi$ is surjective, $\underline{E}$ is said to be *uniformizable*; in this case, its *rank* is defined as the rank of Λ_ϕ as an A-module.

The following is a well-known lemma (see e.g. [Gos98]Lemma 5.9.12).

Lemma 2.2.6. *The A-module structure of $\mathrm{Lie}(E)$ induced by $\mathrm{Lie}(\phi)$ can be extended to a structure of topological vector space over K_∞.*

Proof. Since the endomorphisms $\{\mathrm{Lie}(\phi_a)\}_{a\in A\setminus\{0\}}$ commute and are invertible, the ring homomorphism $\Psi : A \to \mathrm{End}_{\mathbb{C}_\infty}(\mathrm{Lie}(E))$ sending a to $\mathrm{Lie}(\phi_a)$ can be extended uniquely to K, and we can fix a basis $\mathrm{Lie}(E) \cong \mathbb{C}_\infty^d$ in which, for all $c \in K$, Ψ_c is a triangular matrix with $N_c := c^{-1}\Psi_c - \mathrm{Id}_d$ nilpotent—precisely, $N_c^d = 0$. We define on $\mathrm{End}_{\mathbb{C}_\infty}(\mathbb{C}_\infty^d)$ the norm $|\cdot|$ sending a matrix to the maximum of the norms of its coefficients; since the norm on $\mathbb{C}_\infty$ is nonarchimedean, $|MN| \leq |M|\cdot|N|$ for all $M, N \in \mathrm{End}_{\mathbb{C}_\infty}({\mathbb{C}_\infty}^d)$. To extend continuously Ψ to K_∞, it suffices to prove that the set $\{|c^{-1}\Psi_c|\}_{c\in K\setminus\{0\}}$ is bounded, so that $|\Psi_c|$ tends to 0 as $\|c\|$ tends to 0.

Since A is a finitely generated $\mathbb{F}_q$-algebra, we can pick a finite set $\{a_1, \dots, a_n\} \subseteq A$ such that the finite products of the a_i's generate A as an $\mathbb{F}_q$-vector space. If we call $M := \max\{1, |N_{a_1}|, \dots, |N_{a_n}|\}$, it's easy to prove that, for all $b \in A$, $|N_b| \leq M^{nd}$. For all $c \in K^\times$, if we write $c = ab^{-1}$ with $a, b \in A \setminus \{0\}$, we have:

$$|c^{-1}\Psi_c| = |a^{-1}\Psi_a(b^{-1}\Psi_b)^{-1}| = \left|(\mathrm{Id}_d + N_a)\left(\sum_{i=0}^{d-1} N_b^i\right)\right| \leq M^{nd^2},$$

which proves the thesis. □

2.2.2 Functor of Anderson eigenvectors

Definition 2.2.7. For any discrete A-module M, its set of *Anderson eigenvectors* is defined as the A-module of continuous A-linear homomorphisms $\mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty)) \subseteq E(\mathbb{C}_\infty)\hat{\otimes} M$, where the A-module structure on $E(\mathbb{C}_\infty)$ is the one induced by ϕ. We denote by $\mathrm{Sf}_\phi : A - \mathbf{Mod} \to A - \mathbf{Mod}$ the natural functor that extends this map.

Remark 2.2.8. Using the identification $E(\mathbb{C}_\infty)\hat{\otimes} M = E(\mathbb{C}_\infty)\overline{\otimes} M$ established in Proposition 2.1.14, we can rewrite:

$$\mathrm{Sf}_\phi(M) = \{\omega \in E(\mathbb{C}_\infty)\hat{\otimes} M | (\phi_a \otimes 1)(\omega) = (1 \otimes a)\omega \ \forall a \in A\}.$$

In particular, $\mathrm{Sf}_\phi(A)$ is precisely the module of special functions as defined in [GM21].

With the following theorem, we describe the functor Sf_ϕ.

Theorem 2.2.9. *Suppose that E is uniformizable. The functor* Sf_ϕ *is naturally isomorphic to* $\mathrm{Hom}_A(\Lambda_\phi^* \otimes_A \Omega, \_)$. *Moreover, the universal object in* $E(\mathbb{C}_\infty)\hat{\otimes}\mathrm{Hom}_A(\Lambda_\phi,\Omega)$ *corresponds to the map* $\widehat{\mathrm{Hom}_A(\Lambda_\phi,\Omega)} \cong {K_\infty\Lambda_\phi}/{\Lambda_\phi} \to E(\mathbb{C}_\infty)$ *sending the projection of* $c \in K_\infty\Lambda_\phi$ *to* $\exp_\phi(c)$.

Proof. Since E is uniformizable, $E(\mathbb{C}_\infty)$ is isomorphic to ${\mathrm{Lie}(E)}/{\Lambda_\phi}$ as a topological A-module. Endow $\mathrm{Lie}(E)$ with the structure of topological K_∞-vector space described in Lemma 2.2.6; the finite K_∞-vector subspace $K_\infty\Lambda_\phi \subseteq \mathrm{Lie}(E)$ admits a topological complement V, which induces an isomorphism of topological A-modules $E(\mathbb{C}_\infty) \cong {K_\infty\Lambda_\phi}/{\Lambda_\phi} \bigoplus V$.

For any discrete A-module M, for any $\omega \in \mathrm{Sf}_\phi(M)$, its projection $\overline{\omega}$ onto $V\hat{\otimes}M$ belongs to $\mathrm{Hom}_A^{cont}(\hat{M}, V)$. Since $\hat{M}$ is compact, the image of $\overline{\omega}$ must be a compact sub-A-module of V; on the other hand, since V is a topological K_∞-vector space, for any $v \in V \setminus \{0\}$ the set $A \cdot v$ is unbounded, so the only compact sub-A-module of V is $\{0\}$. We deduce that, for any $\omega \in \mathrm{Sf}_\phi(M)$, $\overline{\omega} = 0$, therefore we have the following natural isomorphisms:

$$\mathrm{Sf}_\phi(M) \cong \mathrm{Hom}_A^{cont}\left(\hat{M}, {K_\infty\Lambda_\phi}/{\Lambda_\phi} \oplus V\right) = \mathrm{Hom}_A^{cont}\left(\hat{M}, {K_\infty\Lambda_\phi}/{\Lambda_\phi}\right);$$

by Lemma 2.1.6, the right hand side is naturally isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(\Lambda_\phi,\Omega), M)$.

Setting $M := \mathrm{Hom}_A(\Lambda_\phi,\Omega)$, and following the identity along the chain of isomorphisms, we deduce that the universal object in $\omega_\phi \in \mathrm{Hom}_A^{cont}\left({K_\infty\Lambda_\phi}/{\Lambda_\phi}, E(\mathbb{C}_\infty)\right)$ is the continuous A-linear map sending the projection of $c \in K_\infty\Lambda_\phi$ to $\exp_\phi(c)$. □

For the sake of completeness, let's prove a statement which does not assume uniformizability.

Theorem 2.2.10. *If we restrict the functor* Sf_ϕ *to the subcategory of torsionless A-modules, it is naturally isomorphic to* $\mathrm{Hom}_A(\mathrm{Hom}_A(\Lambda_\phi,\Omega), \_)$.

Moreover, the universal object in $E(\mathbb{C}_\infty)\hat{\otimes}\mathrm{Hom}_A(\Lambda_\phi,\Omega)$ *corresponds to the map*

$$\widehat{\mathrm{Hom}_A(\Lambda_\phi,\Omega)} \cong {K_\infty\Lambda_\phi}/{\Lambda_\phi} \to E(\mathbb{C}_\infty)$$

sending the projection of $c \in K_\infty\Lambda_\phi$ *to* $\exp_\phi(c)$.

Proof. The map $\exp_\phi$ is open because its Jacobian at all points is the identity; call C its image. Since C is an open $\mathbb{F}_q$-vector space, the quotient ${E(\mathbb{C}_\infty)}/{C}$ is a discrete A-module.

A discrete A-module M is torsionless if and only if it has no nontrivial compact submodules; in this case, $\hat{M}$ is a compact A-module with no nontrivial discrete quotients. In particular, for any function $f \in \mathrm{Sf}_\phi(M) = \mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty))$, its projection onto ${E(\mathbb{C}_\infty)}/{C}$ is trivial, hence the image of f must be contained in C. The rest of the proof is the same as Theorem 2.2.9 up to substituting $E(\mathbb{C}_\infty)$ with C. □

Definition 2.2.11. We define the *universal Anderson eigenvector* $\omega_\phi \in \mathbb{C}_\infty\hat{\otimes}\mathrm{Hom}_A(\Lambda_\phi,\Omega)$ as the universal object of the functor Sf_ϕ.

As a corollary, we can describe the isomorphism class of the module of special functions $\mathrm{Sf}_\phi(A)$ for any Anderson A-module E, as already done by Gazda and Maurischat ([GM21][Thm. 3.11]).

Corollary 2.2.12. *The following isomorphism of A-modules holds:*

$$\mathrm{Sf}_\phi(A) = \{\omega \in E(\mathbb{C}_\infty)\hat{\otimes}A | (\phi_a \otimes 1)(\omega) = (1 \otimes a)\omega \ \forall a \in A\} \cong \mathrm{Hom}_A(\Omega, \Lambda_\phi).$$

Remark 2.2.13. Fix an $\mathbb{F}_q$-basis $(\mu_i)_i$ of the discrete A-module $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$, with $(\mu_i^*)_i$ dual basis of ${}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$. By Proposition 2.1.14 we can express the universal object in the following alternative way as an element of $E(\mathbb{C}_\infty)\hat{\otimes}\,\mathrm{Hom}_A(\Lambda_\phi, \Omega)$:

$$\omega_\phi = \sum_i \exp_\phi(\mu_i^*) \otimes \mu_i,$$

where by slight abuse of notation we considered $\exp_\phi$ as a map from ${}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$ to $E(\mathbb{C}_\infty)$.

2.3 Proof of a conjecture of Gazda and Maurischat

Recall from the introduction that an Anderson A-module $\underline{E} = (E, \phi)$ of dimension 1 is called a *Drinfeld module*. For simplicity, we assume $E = \mathbb{G}_a$, so that $E(\mathbb{C}_\infty) = \mathbb{C}_\infty$.

Denote by $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty$ the Frobenius endomorphism. Recall from Section 1.3 that we can think of the action of A on E as a ring homomorphism $\phi : A \to \mathbb{C}_\infty[\tau]$ sending a to $\phi_a = \sum_i a_i \tau^i$, and that the *rank* of $\underline{E}$ is defined as the unique positive integer r such that $r \cdot \deg(a) = \deg_\tau(\phi_a)$.

We apply the results of Section 2.2 in the context of a Drinfeld module $\underline{E} = (E, \phi)$ of rank 1 (i.e. such that its period lattice Λ_ϕ has rank 1 as an A-module), with the further assumption that $\infty \in X(\mathbb{F}_q)$, to answer a question posed by Gazda and Maurischat in [GM21].

The following property holds (see [ANT17a][Lemma 3.6, Rmk. 3.10] or [GM21][Prop. 3.18]).

Proposition 2.3.1. *There exists an element $f_\phi \in \mathrm{Frac}(A_{\mathbb{C}_\infty})$ such that, for all $\omega \in \mathrm{Sf}_\phi(A) \subseteq \mathbb{C}_\infty \hat{\otimes} A$, we have:*

$$(\tau \otimes 1)\omega = f_\phi \omega.$$

Moreover, for all $x \in \mathbb{C}_\infty \hat{\otimes} A$, if $(\tau \otimes 1)x = f_\phi x$ then x belongs to $\mathrm{Sf}_\phi(A)$.

Remark 2.3.2. The element f_ϕ is a fundamental object in the study of a Drinfeld module $(\mathbb{G}_a, \phi)$ of rank 1, and is called *shtuka function*; the previous proposition can be seen as an alternative to its usual definition (see [Tha93], [Gos98][Def. 7.11.2]). In this thesis, we study f_ϕ and related functions in Chapter 4 (see Definition 4.1.10).

In particular, if there is some $\omega \in \mathrm{Sf}_\phi(A)$ which is an invertible element of the ring $\mathbb{C}_\infty \hat{\otimes} A$, for all $\omega' \in \mathrm{Sf}_\phi(A)$ we have $(\tau \otimes 1)\left(\frac{\omega'}{\omega}\right) = \frac{\omega'}{\omega}$, i.e. $\frac{\omega'}{\omega} \in \mathbb{F}_q \otimes_{\mathbb{F}_q} A$, hence $\mathrm{Sf}_\phi(A) = A \cdot \omega$.

The conjecture of Gazda and Maurischat in [GM21] is about the converse statement.

Conjecture 2.3.3 ([GM21][Question])**.** *If $\mathrm{Sf}_\phi(A) \cong A$, there is some $\omega \in \mathrm{Sf}_\phi(A)$ which is invertible as an element of $\mathbb{C}_\infty \hat{\otimes} A$.*

We answer affirmatively with Theorem 2.3.7.

First, we prove two results to show that Pontryagin duality is well-behaved with respect to norms. For starters, we endow the space $\widehat{K_\infty} \cong \Omega \otimes_A K_\infty$ with a norm $|\cdot|$ such that it is a normed vector space over $(K_\infty, \|\cdot\|)$, and for any ideal $J < A$ we use the same notation for the induced norm on the quotient $\hat{J}$ of $\widehat{K_\infty}$; note that, since $\widehat{K_\infty}$ has dimension 1 as a K_∞-vector space, $|\cdot|$ is unique up to a scalar factor in $\mathbb{R}_{>0}$.

Proposition 2.3.4. *Up to a scalar factor in $\mathbb{R}_{>0}$, for all $f \in \widehat{K_\infty} \setminus \{0\}$, we have*

$$|f|^{-1} = \min\{\|\lambda\| \text{ s.t. } \lambda \in K_\infty \text{ and } f(\lambda) \neq 0\}.$$

Proof. Let $t \in K_\infty$ be a uniformizer: we can identify K_∞ with $\mathbb{F}_q((t))$, where if the series $p(t) = \sum_{i\in\mathbb{Z}} \lambda_i t^i \in \mathbb{F}_q((t))$ has leading term $\lambda_k t^k$, its norm is q^{-k}. Consider the function $dt \in \widehat{\mathbb{F}_q((t))}$ which sends $p(t)$ as defined above to λ_{-1}: under the identification $K_\infty = \mathbb{F}_q((t))$, we have $\widehat{K_\infty} = \mathbb{F}_q((t))dt$, and up to a scalar factor in $\mathbb{R}_{>0}$ we can assume $|dt| = q^{-1}$.

Take $\mu \in \mathbb{F}_q((t))dt \setminus \{0\}$ with leading term $b_k t^k dt$, so that $|\mu| = q^{-k-1}$: if $p \in \mathbb{F}_q((t))$ has $\|p\| < q^{k+1}$, its leading term has degree at least $-k$, hence $\mu(p) = 0$; on the other hand $\|t^{-k-1}\| = q^{k+1}$ and $\mu(t^{-k-1}) = b_k \neq 0$. In particular:

$$|\mu|^{-1} = q^{k+1} = \min\{\|p\| \text{ s.t. } p \in \mathbb{F}_q((t)) \text{ and } \mu(p) \neq 0\}. \qquad \square$$

Proposition 2.3.5. *Let $J < A$ be a nonzero ideal and fix an $\mathbb{F}_q$-basis $(a_i)_{i\in I}$ of J strictly ordered by degree, with $(a_i^*)_{i\in I}$ dual basis of $\hat{J}$. The sequence $(|a_i^*|)_{i\in I}$ is strictly decreasing.*

Proof. We can assume $I \subseteq \mathbb{Z}$ to be the set of degrees of elements in J, and that a_i has degree i for all $i \in I$. For all $i \in I$ set $b_i := a_i$, while for all $i \in \mathbb{Z} \setminus I$ choose some $b_i \in K_\infty$ with valuation $-i$: since all nonzero elements of K_∞ have integer valuation, it's easy to check that every $c \in K_\infty$ can be expressed in a unique way as $\sum_{i\in\mathbb{Z}} \lambda_i b_i$ where $\lambda_i \in \mathbb{F}_q$ for all $i \in \mathbb{Z}$ and $\lambda_i = 0$ for $i \gg 0$. Denote by $(b_i^*)_{i\in\mathbb{Z}}$ the sequence in $\widehat{K_\infty}$ determined by the property $b_i^*(b_j) = \delta_{i,j}$ for all $i, j \in \mathbb{Z}$. By Proposition 2.3.4, up to rescaling $|\cdot|$ by some positive real factor, we have for all $i \in \mathbb{Z}$:

$$|b_i^*|^{-1} = \min\{\|c\| \text{ s.t. } c \in K_\infty \text{ and } b_i^*(c) \neq 0\} = \min\left\{\left\|\sum_{j\in\mathbb{Z}} \lambda_j b_j\right\| \text{ s.t. } \lambda_i \neq 0\right\} = \|b_i\|.$$

Let's prove that any $c \in \widehat{K_\infty}$ can be expressed in a unique way as a series $\sum_{i\in\mathbb{Z}} \lambda_i b_i^*$ with $\lambda_i \in \mathbb{F}_q$ for all i and $\lambda_i = 0$ for $i \ll 0$. We have:

$$c = \sum_{i\in\mathbb{Z}} \lambda_i b_i^* \Leftrightarrow c(b_j) = \left(\sum_{i\in\mathbb{Z}} \lambda_i b_i^*\right)(b_j) \forall j \in \mathbb{Z} \Leftrightarrow c(b_j) = \lambda_j \forall j \in \mathbb{Z},$$

which proves uniqueness. Viceversa, since c is continuous, $c(b_j) = 0$ for $j \ll 0$, and since the sequence $(|b_j^*|)_{j\in\mathbb{Z}} = (\|b_j\|^{-1})_{j\in\mathbb{Z}}$ is strictly decreasing and tends to 0, the series $\sum_{i\in\mathbb{Z}} c(b_i) b_i^*$ converges in $\widehat{K_\infty}$.

For any $c \in \widehat{K_\infty}$, call $\overline{c}$ its projection onto $\hat{J}$. Since $(b_i)_{i\in I} = (a_i)_{i\in I}$ is an $\mathbb{F}_q$-basis of J, $\overline{b_i^*} = a_i^*$ if $i \in I$, and $\overline{b_i^*} = 0$ otherwise. For all $i \in I$, we have:

$$|a_i^*| = \min\{|c| \text{ s.t. } \overline{c} = a_i^*\} = \min\left\{\left|\sum_{j\in\mathbb{Z}} \lambda_j b_j^*\right| \text{ s.t. } \lambda_j = \delta_{i,j} \forall j \in I\right\} = |b_i^*| = \|a_i\|^{-1}. \qquad \square$$

To prove the main result of this section, we need to use a well known property of the exponential function $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ (see for example [Gos98][Section 4.2]).

Proposition 2.3.6. *Let $(\mathbb{G}_a, \phi)$ be a Drinfeld module. The following identity holds for all $z \in \mathbb{C}_\infty$:*

$$\exp_\phi(z) = z \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right).$$

Theorem 2.3.7. *Suppose* $\mathrm{Sf}_\phi(A) \cong A$*. Then, there is a special function in* $\mathrm{Sf}_\phi(A)$ *which is invertible as an element of* $\mathbb{C}_\infty \hat{\otimes} A$*.*

Proof. As shown in Corollary 2.2.12, $\Lambda_\phi \cong \Omega$. Fix an $\mathbb{F}_q$-basis $(a_i)_{i\in I}$ of A like in the proof of Proposition 2.3.5, with $a_0 = 1$, and let $(a_i^*)_{i\in I}$ be the dual basis of its Pontryagin dual $\hat{A} \cong {}^{\Omega \otimes_A K_\infty}\!/_{\Omega} \cong {}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$.

By Remark 2.2.13, we can write the universal Anderson eigenvector as an infinite series $\omega_\phi = \sum_i \exp_\phi(a_i^*) \otimes a_i \in \mathbb{C}_\infty \hat{\otimes} A$ (where by slight abuse of notation we considered $\exp_\phi$ as a map from ${}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$ to $\mathbb{C}_\infty$). To prove it is invertible, it suffices to show that, for all $i \geq 1$, $\|\exp_\phi(a_0^*)\| > \|\exp_\phi(a_i^*)\|$: indeed, if this is the case, and we set $\omega := (\exp_\phi(a_0^*)^{-1} \otimes 1)\omega_\phi$, the element $1 - \omega \in \mathbb{C}_\infty \hat{\otimes} A$ has norm less than 1, hence the series $\sum_{n\geq 0}(1-\omega)^n$ converges in $\mathbb{C}_\infty \hat{\otimes} A$, and is an inverse to $1 - (1-\omega) = \omega$.

For all indices i, choose a lifting $c_i \in K_\infty \Lambda_\phi \subseteq \mathbb{C}_\infty$ of $a_i^* \in {}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$ with the least norm, so that $\|c_i\| = |a_i^*|$; in particular, since Λ_ϕ has rank 1, there are no $\lambda \in \Lambda_\phi$ such that $\|\lambda\| = \|c_i\|$, so we have:

$$\|\exp_\phi(a_i^*)\| = \|c_i\| \prod_{\lambda \in \Lambda_\phi \setminus \{0\}} \left\| 1 - \frac{c_i}{\lambda} \right\| = \|c_i\| \prod_{\substack{\lambda \in \Lambda_\phi \setminus \{0\} \\ \|\lambda\| \leq \|c_i\|}} \left\| 1 - \frac{c_i}{\lambda} \right\| = \|c_i\| \prod_{\substack{\lambda \in \Lambda_\phi \setminus \{0\} \\ \|\lambda\| < \|c_i\|}} \left\| \frac{c_i}{\lambda} \right\|.$$

Since by Proposition 2.3.5 the sequence $(\|c_i\|)_i$ is strictly decreasing, from the previous equality we deduce that the sequence $(\|\exp_\phi(a_i^*)\|)_i$ is also strictly decreasing. In particular, $\|\exp_\phi(a_0^*)\| > \|\exp_\phi(a_i^*)\|$ for all $i \geq 1$. □

Chapter 3

A topological approach to the convergence of rational functions on X_{K_∞}

In this chapter, $X, A, K_\infty, \mathbb{C}_\infty$ are defined as in Section 1.3. Consider the d-th symmetric power $X^{[d]}$ for some positive integer d. For all field extensions $L/\mathbb{F}_q$, $X^{[d]}(L)$ is the set of effective L-rational divisors on X of degree d, and for all $D \in X^{[d]}(L)$, we denote by $H^0(X_L, D)$ the global sections of the line bundle on X_L associated to D. For any effective divisor $D \in X^{[d]}(\mathbb{F}_{q^e})$, for any finite field extension L/K_∞, we endow the space $H^0(X_L, D)$ with the natural topology of finite vector space over L. The aim of this section is to endow $X^{[d]}(L)$ with a topology, which we call "natural compact topology" (see Definition 3.1.3), such that the following proposition holds.

Proposition (Prop. 3.2.9)**.** *Fix a finite field extension L/K_∞ and an effective divisor D_- in $X^{[d]}(\mathbb{F}_{q^e})$, and consider a sequence $(h_m)_m$ in $H^0(X_L, D_-)$.*

If the sequence $(\mathrm{Div}(h_m) + D_-)_m$ converges to $D_+ \in X^{[d]}(L)$ in the natural compact topology, there are $(\lambda_m)_m$ in $L^\times$ such that the sequence $(\lambda_m h_m)_m$ converges in $H^0(X_L, D_-)$ to some nonzero h with $\mathrm{Div}(h) = D_+ - D_-$.

If the sequence $(h_m)_m$ converges in $H^0(X_L, D_-)$ to some nonzero h, the sequence $(\mathrm{Div}(h_m)+D_-)_m$ converges to $\mathrm{Div}(h) + D_- \in X^{[d]}(L)$ in the natural compact topology.

We need a topology on the L-points of other projective $\mathbb{F}_q$-schemes (such as the powers cartesian powers X^d for $d \geq 1$ and the Jacobian variety $\mathcal{A}$ of X). To ensure their good interaction we prove that the compact topology that we define is functorial in Proposition 3.1.5.

3.1 Natural compact topology on K_∞-rational points of $\mathbb{F}_q$-schemes

Through this section, L is a finite field extension of K_∞ with residue field $\mathbb{F}_L \subseteq \mathcal{O}_L$ and $\mathfrak{m}_L \subseteq \mathcal{O}_L$ maximal ideal, and Y is a proper $\mathcal{O}_L$-scheme. We aim to construct a functor from proper schemes over $\mathcal{O}_L$ to compact Hausdorff topological spaces, sending Y to $Y(\mathcal{O}_L) = Y(L)$.

Lemma 3.1.1. *The natural maps $\mathrm{red}_{L,k} : Y(\mathcal{O}_L) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ for all $k \geq 1$, where we omit the dependence on Y, induce a bijection $Y(\mathcal{O}_L) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k)$.*

Proof. Since $\mathrm{Spec}(\mathcal{O}_L) \cong \varinjlim_k \mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k)$, we have:

$$\begin{aligned} Y(\mathcal{O}_L) &\cong \mathrm{Hom}_{\mathcal{O}_L}\left(\varinjlim_k \mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k), Y\right) \\ &\cong \varprojlim_k \mathrm{Hom}_{\mathcal{O}_L}\left(\mathrm{Spec}(\mathcal{O}_L/\mathfrak{m}_L^k), Y\right) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k). \end{aligned}$$ □

Remark 3.1.2. If we endow the spaces $Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ with the discrete topology, the limit topology induced on $Y(\mathcal{O}_L) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ is Hausdorff. Since Y is finite-type over $\mathcal{O}_L$ and $\mathcal{O}_L/\mathfrak{m}_L^k$ is finite for all k, the space $Y(\mathcal{O}_L/\mathfrak{m}_L^k)$ is finite for all k, so the limit topology makes $Y(\mathcal{O}_L)$ into a compact space. Moreover, $Y(\mathcal{O}_L)$ can be endowed with an ultrametric distance $\bar{d}$ as follows:

$$\bar{d}(P,Q) := \max_{k\in\mathbb{N}} \left\{ \frac{1}{p^k} \,\middle|\, \mathrm{red}_{L,k}(P) \neq \mathrm{red}_{L,k}(Q) \right\}.$$

The only non obvious property to check is the ultrametric inequality: for all $P, Q, R \in Y(\mathcal{O}_L)$, if $\mathrm{red}_{L,k}(P) = \mathrm{red}_{L,k}(Q)$ and $\mathrm{red}_{L,k}(Q) = \mathrm{red}_{L,k}(R)$ we have $\mathrm{red}_{L,k}(P) = \mathrm{red}_{L,k}(R)$, hence

$$\bar{d}(P,R) \leq \max\{\bar{d}(P,Q), \bar{d}(Q,R)\}.$$

Definition 3.1.3. We call *natural compact topology* the topology induced on $Y(L) = Y(\mathcal{O}_L)$ by the bijection $Y(\mathcal{O}_L) \cong \varprojlim_k Y(\mathcal{O}_L/\mathfrak{m}_L^k)$.

Definition 3.1.4. We denote by $\mathrm{red}_L : Y(\mathcal{O}_L) \to Y(\mathcal{O}_L)$ and call *reduction* the composition of the map $\mathrm{red}_{L,1} : Y(\mathcal{O}_L) \to Y(\mathbb{F}_L)$, induced by the projection $\mathcal{O}_L \to \mathbb{F}_L$, and the map $Y(\mathbb{F}_L) \hookrightarrow Y(\mathcal{O}_L)$ induced by the inclusion $\mathbb{F}_L \subseteq \mathcal{O}_L$.

From this point onwards, unless otherwise stated, we interpret the set $Y(L)$ as endowed with the natural compact topology. Similarly, if Y' is a proper $\mathbb{F}_q$-scheme, the set $Y'(L) = Y'_{\mathcal{O}_L}(L)$ is always endowed with the natural compact topology.

Proposition 3.1.5. *The map associating to a proper $\mathcal{O}_L$-scheme Y the topological space $Y(\mathcal{O}_L)$ can be extended to a functor F_L.*

Proof. For every morphism $\varphi : Z \to Y$ of proper $\mathcal{O}_L$-schemes, the induced map $\varphi_{\mathcal{O}_L} : Z(\mathcal{O}_L) \to Y(\mathcal{O}_L)$ induces a system of maps $(\varphi_{\mathcal{O}_L/\mathfrak{m}_L^k} : Z(\mathcal{O}_L/\mathfrak{m}_L^k) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k))_k$ which commute with the transition maps of the diagrams $(Z(\mathcal{O}_L/\mathfrak{m}_L^k))_k$ and $(Y(\mathcal{O}_L/\mathfrak{m}_L^k))_k$, hence $\varphi_{\mathcal{O}_L}$ is continuous.

If we set $F_L(\varphi) := \varphi_{\mathcal{O}_L}$ for all morphisms, it's easy to check that F_L sends the identity map to the identity map and preserves composition, hence it is a functor. □

Remark 3.1.6. We also obtain a functor from proper $\mathbb{F}_q$-schemes to topological spaces, sending a scheme Y to $Y(\mathcal{O}_L) = Y(L)$, by precomposing F_L with the base change $Y \mapsto Y_{\mathcal{O}_L}$.

Lemma 3.1.7. *Let $f : Z \to Y$ be a morphism of proper $\mathcal{O}_L$-schemes. Fix a subset $V \subseteq Y(L)$ with preimage $U \subseteq Z(L)$, such that $F_L(f)|_U : U \to V$ is bijective. Then $F_L(f)|_U$ is a homeomorphism.*

Proof. The map $F_L(f) : Z(L) \to Y(L)$ is closed, being a continuous map between compact Hausdorff spaces. Any closed set of U can be written as $C \cap U$, with $C \subseteq Z(L)$ closed. We have:

$$F_L(f)(C \cap U) = F_L(f)\left(C \cap F_L(f)^{-1}(V)\right) = F_L(f)(C) \cap V,$$

which is closed in V because $F_L(f)(C)$ is closed in $Y(L)$. This means that $F_L(f)|_U$ is closed, and since it induces a bijection between U and V, it is a homeomorphism. □

Remark 3.1.8. In the case of the projective space $\mathbb{P}^n$ of dimension n over $\mathbb{F}_q$, the set $\mathbb{P}^n(L)$ is in bijection with $(L^{n+1} \setminus \{0\})/L^\times$; since the latter has a natural topology induced by L, the former also does, and it's easy to check that it's the same as the natural compact topology we defined.

The following statements show that the functor F_L sends group schemes to topological groups.

Lemma 3.1.9. *The topological spaces $F_L(Y \times_{\mathcal{O}_L} Y)$ and $F_L(Y) \times F_L(Y)$ are naturally homeomorphic.*

Proof. The projections $\pi_1, \pi_2 : Y \times Y \to Y$ induce a natural continuous map from $F_L(Y \times_{\mathcal{O}_L} Y)$ to $F_L(Y) \times F_L(Y)$. Since both spaces are compact and Hausdorff, the map is closed; since the underlying function is the natural bijection $(Y \times_{\mathcal{O}_L} Y)(L) \cong Y(L) \times Y(L)$, the map is a homeomorphism. □

Proposition 3.1.10. *If Y is a (commutative) group scheme over $\mathcal{O}_L$, the metric on $Y(L)$ is translation invariant, and makes it into a (commutative) topological group.*

Proof. By Lemma 3.1.9, we identify $F_L(Y \times_{\mathcal{O}_L} Y) \cong F_L(Y) \times F_L(Y)$ via a natural homeomorphism.

Call e the identity, i the inverse, and m the multiplication of Y. Then $F_L(Y)$ has a natural structure of topological group, with identity $F_L(e)$, inverse $F_L(i)$ and multiplication $F_L(m)$, because all the necessary diagrams commute by functoriality. For the same reason, if Y is commutative as a group scheme, $Y(L)$ is commutative as a topological group.

To prove the invariance of the metric, we need to show that every translation is an isometry. Fix a morphism of $\mathcal{O}_L$-schemes $P : \mathrm{Spec}(\mathcal{O}_L) \to Y$ (i.e. $P \in Y(\mathcal{O}_L)$), and consider the following:

$$l_P : Y \cong \mathrm{Spec}(\mathcal{O}_L) \times_{\mathcal{O}_L} Y \xrightarrow{P \times id_Y} Y \times_{\mathcal{O}_L} Y \xrightarrow{m} Y,$$

so that $F_L(l_P) : Y(L) \to Y(L)$ is the left translation by P. It's immediate to check that, if we call $-P$ the inverse of P in $Y(\mathcal{O}_L)$, l_{-P} is the two-sided inverse of l_P, therefore they are isomorphisms. In particular l_P induces a family of bijections $\{Y(\mathcal{O}_L/\mathfrak{m}_L^k) \to Y(\mathcal{O}_L/\mathfrak{m}_L^k)\}_{k \geq 1}$, whose limit is precisely $F_L(l_P)$, hence $F_L(l_P)$ is an isometry. The proof for right translations is essentially the same. □

Corollary 3.1.11. *Suppose that Y is a commutative group scheme. Denote by addition the group law on $Y(L)$ and by 0 its identity element. If $(P_i)_{i \in \mathbb{N}}$ is a sequence in $Y(L)$ converging to 0, then the series $\sum_i P_i$ is a well defined element of $Y(L)$ (i.e. the sequence of partial sums converge).*

Proof. Call $\bar{d}$ the distance on $Y(L)$. Since $\bar{d}$ is ultrametric, we just need the limit of the distances $\bar{d}(S_k, S_{k-1})$ to be 0, where $S_k := \sum_{i=0}^k P_i$. Since the metric is translation invariant, $\lim_k \bar{d}(S_k, S_{k-1}) = \lim_k \bar{d}(P_k, 0)$, which is zero by hypothesis. □

3.2 The natural compact topology on the space of divisors

In this section we state some propositions about the symmetric powers of a curve and its Jacobian. Most results are already stated and proven in [Mil86].

Recall the definition of X; S_d is the permutation group of d elements. We have the following (see [Mil86][Prop. 3.1, Prop. 3.2]).

Proposition 3.2.1. *Fix a positive integer d. Consider the natural right action of S_d on X^d and call its quotient $X^{[d]}$. Then $X^{[d]}$ is a proper smooth $\mathbb{F}_q$-scheme.*

The following result (see [Mil86][Thm. 3.13]) gives us the functorial interpretation of the symmetric power $X^{[d]}$.

Theorem 3.2.2. *Consider the functor* Div_X^d *which sends an* $\mathbb{F}_q$*-algebra* R *to the set of relative effective Cartier divisors of degree* d *on* X_R *over* R *(i.e. effective Cartier divisors on* X_R *which are finite and flat of rank* d *over* R*). This functor is represented by* $X^{[d]}$*.*

Corollary 3.2.3. *For every field extension* $E/\mathbb{F}_q$*,* $X^{[d]}(E)$ *is in bijection with the finite closed* E*-subschemes of* X_E *of degree* d*.*

Let's continue with the fundamental property of the Jacobian variety (see [Mil86][Thm. 1.1]). For any $\mathbb{F}_q$-algebra R, we denote by $\pi_R : X_R \to \mathrm{Spec}(R)$ the structure morphism; for any field extension ${}^{L}\!/\!_{\mathbb{F}_q}$, we denote by $\deg : \mathrm{Pic}(X_L) \to \mathbb{Z}$ the natural function associating to an invertible sheaf—up to isomorphism—its degree.

Theorem 3.2.4. *Call* P_X^0 *the natural functor from* $\mathbb{F}_q$*-algebras to abelian groups such that for any* $\mathbb{F}_q$*-algebra* R*:*

$$P_X^0(R) = \frac{\{\mathcal{L} \in \mathrm{Pic}(X_R) | \deg(i^*\mathcal{L}) = 0 \ \forall i : \mathrm{Spec}(k) \to \mathrm{Spec}(R) \ \textit{closed point}\}}{\pi_R^*(\mathrm{Pic}(\mathrm{Spec}(R))}.$$

There is an abelian variety $\mathcal{A}$ *over* $\mathbb{F}_q$*, called the Jacobian variety of* X*, and a natural transformation of functors* $P_X^0 \to \mathcal{A}$ *which induces an isomorphism* $P_X^0(R) \cong \mathcal{A}(R)$ *whenever* $X(R) \neq \emptyset$*.*

Let's fix a point $\infty' \in X(\mathbb{F}_{q^e})$ with support at ∞. The following result clarifies the relation between the symmetric powers of X and $\mathcal{A}$ (see [Mil86][Thm. 5.2]).

Theorem 3.2.5. *For all* $d \geq 1$*, the point* $\infty' \in X_{\mathbb{F}_{q^e}}$ *induces a natural morphism of* $\mathbb{F}_{q^e}$*-schemes* $J^d : X_{\mathbb{F}_{q^e}}^{[d]} \to \mathcal{A}_{\mathbb{F}_{q^e}}$*. Moreover, the morphism* $J^g : X_{\mathbb{F}_{q^e}}^{[g]} \to \mathcal{A}_{\mathbb{F}_{q^e}}$ *is birational and surjective.*

Remark 3.2.6. For every field $E/\mathbb{F}_{q^e}$, at the level of E-points the morphism J^d sends an effective divisor D of degree d to the class of $D - d\infty'$.

Finally, we give a result on the fibers of the map J^d (see [Mil86][Rmk. 5.6.(c)] and [Har77][Prop. II.7.12]).

Proposition 3.2.7. *Fix a field extension* $E/\mathbb{F}_{q^e}$ *and a point* $D \in X_{\mathbb{F}_{q^e}}^{[d]}(E)$*, define* $P := J^d \circ D \in \mathcal{A}_{\mathbb{F}_{q^e}}(E)$*, and call* V *the* E*-vector space* $H^0(X_E, D)$*. The fiber* $(J^d)^*P$ *is naturally isomorphic as an* E*-scheme to* $\mathbb{P}(V)$*.*

For any field extension E'/E*, for all* $f \in E' \otimes_E V \cong H^0(X_{E'}, D)$*, the isomorphism sends the line* $E' \cdot f \in \mathbb{P}(V)(E')$ *to* $\mathrm{Div}(f) + D \in X^{[d]}(E')$*.*

Corollary 3.2.8. *Let* $D \in X_{\mathbb{F}_{q^e}}^{[d]}(\overline{E})$ *with* $h^0(D) = 1$*. If* $J^d \circ D \in \mathcal{A}_{\mathbb{F}_{q^e}}(\overline{E})$ *factors through some* $P \in \mathcal{A}_{\mathbb{F}_{q^e}}(E)$*,* D *factors through some* $D' \in X_{\mathbb{F}_{q^e}}^{[d]}(E)$*.*

Proof. Since D factors through some finite extension $\Phi : \mathrm{Spec}(E') \to \mathrm{Spec}(E)$, we can assume $D \in X_{\mathbb{F}_{q^e}}^{[d]}(E')$ without loss of generality. By Proposition 3.2.7, the pullback of $P \circ \Phi \in \mathcal{A}_{\mathbb{F}_{q^e}}(E')$ along J^d is a morphism $\mathrm{Spec}(E') \to X_{\mathbb{F}_{q^e}}^{[d]}$, hence it is exactly D. If $Z \to X_{\mathbb{F}_{q^e}}^{[d]}$ is the pullback of P along J^d, $Z \times_{\mathrm{Spec}(E)} \mathrm{Spec}(E')$ is isomorphic to $\mathrm{Spec}(E')$; we deduce that $Z \cong \mathrm{Spec}(E)$, and D factors through $Z \to X_{\mathbb{F}_{q^e}}^{[d]}$. □

With the following proposition we can finally switch between convergence of functions and convergence of divisors, an essential step to prove the functional identities in Theorem 4.1.9 and Theorem 4.2.1.

Proposition 3.2.9. *Fix a finite field extension L/K_∞ and an effective divisor D_- in $X^{[d]}(\mathbb{F}_{q^e})$, and consider a sequence $(h_m)_m$ in $H^0(X_L, D_-)$.*

If the sequence $(\mathrm{Div}(h_m) + D_-)_m$ converges to $D_+ \in X^{[d]}(L)$, there are $(\lambda_m)_m$ in $L^\times$ such that $(\lambda_m h_m)_m$ converges in $H^0(X_L, D_-)$ to some nonzero h with $\mathrm{Div}(h) = D_+ - D_-$.

If the sequence $(h_m)_m$ converges in $H^0(X_L, D_-)$ to some nonzero h, the sequence $(\mathrm{Div}(h_m)+D_-)_m$ converges to $\mathrm{Div}(h) + D_- \in X^{[d]}(L)$.

Proof. Call $V := H^0(X_{\mathbb{F}_{q^e}}, D_-)$ and call Z_d the pullback of the closed subscheme $[D_- - d\infty'] \in \mathcal{A}_{\mathbb{F}_{q^e}}$ along $J^d : X^{[d]}_{\mathbb{F}_{q^e}} \to \mathcal{A}_{\mathbb{F}_{q^e}}$, so that $\mathrm{Div}(h_m) + D_- \in Z_d(L)$ for all m. As we noted in Remark 3.1.8, $\mathbb{P}(V)(L)$ is homeomorphic to $(H^0(X_L, D_-) \setminus \{0\})/L^\times$ endowed with the quotient topology. On the other hand, by Proposition 3.2.7 (setting $E = \mathbb{F}_{q^e}$ and $D = D_-$), the $\mathbb{F}_{q^e}$-schemes $\mathbb{P}(V)$ and Z_d are isomorphic; in particular, the induced map $\mathbb{P}(V)(L) \to Z_d(L)$, which sends a line $L \cdot f \in H^0(X_L, D_-)$ to $\mathrm{Div}(f) + D_-$, is a homeomorphism in the natural compact topology, by Remark 3.1.6.

If the sequence $(\mathrm{Div}(h_m) + D_-)_m$ converges to $D_+ \in Z_d(L)$, this proves that the equivalence classes $([h_m])_m$ in $H^0(X_L, D_-)/L^\times$ do converge to an equivalence class $[h]$ whose divisor is $D_+ - D_-$. Since the projection is open, we can lift this convergence to $H^0(X_L, D_-)$ up to scalar multiplication.

The map $H^0(X_L, D_-) \setminus \{0\} \to Z(L)$ sending a function f to the effective divisor $\mathrm{Div}(f) + D_-$ is continuous. In particular, if the sequence $(h_m)_m$ converges to a nonzero $h \in H^0(X_L, D_-)$, the sequence $(\mathrm{Div}(h_m) + D_-)_m$ converges to $\mathrm{Div}(h) + D_- \in Z_d(L)$. □

3.3 Frobenius and divisors

Fix a nonzero ideal $I < A$, denote by $\bar{I}$ its class in the ideal class group $Cl(A)$; with slight abuse of notation, call I also the corresponding effective divisor of X (if e.g. $I = A$, the corresponding divisor is the unique divisor of degree 0). Call $\Xi \in X(K)$ the morphism $\mathrm{Spec}(K) \to X \setminus \{\infty\}$ corresponding to the canonical inclusion $A \hookrightarrow K$. From now on, we assume ∞ to be $\mathbb{F}_q$-rational.

In Subsection 3.3.1, we recall the notion of Frobenius twist $P^{(1)}$ for a point P in $X^{[d]}(K_\infty)$, and study its behavior with respect to the natural compact topology. The main result is Proposition 3.3.6, where we prove that the sequence $(P^{(m)})_m$ converges to $\mathrm{red}_{K_\infty}(P)$ in $X^{[d]}(K_\infty)$ (with the natural compact topology).

In Subsection 3.3.2, we study the divisor of a rational function h on X_{K_∞} with respect to its expansion $\sum_{i \geq k} c_i u^i$ as an element of $K((u))$, where u is a uniformizer of K_∞ . Among several useful results, the most significant is Proposition 3.3.16, where we state the identity $\mathrm{Div}(c_k) = \mathrm{red}_{K_\infty}(\mathrm{Div}(h))$.

Finally, in Subsection 3.3.3, we construct the divisors $V_{\bar{I},*,m}$ for $m \gg 0$ and $V_{\bar{I},*}$ in $X^{[g]}(K_\infty)$ (see Proposition 3.3.25), uniquely defined by the following linear equivalences for $m \gg 0$:

$$\begin{cases} V_{\bar{I},*,m} - V_{\bar{I},*,m}^{(1)} \sim \Xi^{(m)} - \Xi^{(1)} \\ \mathrm{red}_{K_\infty}(V_{\bar{I},*,m}) \sim (\deg(I) + g)\infty - I \end{cases} ; \qquad \begin{cases} V_{\bar{I},*} - V_{\bar{I},*}^{(1)} \sim \infty - \Xi \\ \mathrm{red}_{K_\infty}(V_{\bar{I},*}) \sim (\deg(I) + g)\infty - I \end{cases} .$$

The main result is the convergence of the sequence $(V_{\bar{I},*,m})_{m \gg 0}$ to $V_{\bar{I},*}^{(1)}$ in $X^{[g]}(K_\infty)$ (Proposition 3.3.27).

3.3.1 Frobenius twist

In this subsection we define the Frobenius twist for a (proper) $\mathbb{F}_q$-scheme Y and study its behavior with respect to the topology of $Y(K_\infty)$. The fundamental results are Proposition 3.3.6 and Lemma 3.3.10.

Definition 3.3.1. Let Y be an $\mathbb{F}_q$-scheme, and R an $\mathbb{F}_q$-algebra. Denote by Frob_R the endomorphism of $\mathrm{Spec}(R)$ induced by raising to the q-th power, and by $F_R^Y := \mathrm{Id}_Y \times \mathrm{Frob}_R$ the endomorphism of $Y_R = Y \times_{\mathrm{Spec}(\mathbb{F}_q)} \mathrm{Spec}(R)$.

Call $\pi_Y : Y_R \to Y$ and $\pi_R : Y_R \to \mathrm{Spec}(R)$ the natural projections. For all $P \in Y(R)$, $\overline{P}$ denotes the unique element of $\mathrm{Hom}_R(\mathrm{Spec}(R), Y_R)$ such that $P = \pi_Y \circ \overline{P}$. We call Frobenius twist of P, denoted by $P^{(1)} \in Y(R)$, the only element such that $\overline{P^{(1)}}$ is the pullback of $\overline{P}$ along F_R^Y. The n-th iteration of the twist is denoted by $P^{(n)}$ for all $n \in \mathbb{N}$.

Lemma 3.3.2. *In the notation of Definition 3.3.1, we have $P^{(1)} = P \circ \mathrm{Frob}_R$.*

Proof. We have the following cartesian diagram:

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "\overline{P^{(1)}}"] \arrow[d, "\mathrm{Frob}_R"'] \arrow[dr, phantom, "\square"] & Y_R \arrow[r, "\pi_R"] \arrow[d, "F_R^Y"] \arrow[dr, phantom, "\square"] & \mathrm{Spec}(R) \arrow[d, "\mathrm{Frob}_R"] \\
\mathrm{Spec}(R) \arrow[r, "\overline{P}"'] & Y_R \arrow[r, "\pi_R"'] & \mathrm{Spec}(R).
\end{tikzcd}$$

Since $\pi_Y \circ F_R^Y = \pi_Y$, $P^{(1)} = \pi_Y \circ \overline{P^{(1)}} = \pi_Y \circ F_R^Y \circ \overline{P^{(1)}} = \pi_Y \circ \overline{P} \circ \mathrm{Frob}_R = P \circ \mathrm{Frob}_R$. $\square$

Remark 3.3.3. In light of Lemma 3.3.2, if Frob_R is an isomorphism, for all $P \in Y(R)$ we can redefine $P^{(k)} \in Y(R)$ as $P \circ (\mathrm{Frob}_R)^k$ for all $k \in \mathbb{Z}$.

Lemma 3.3.4. *Fix a positive integer d, an $\mathbb{F}_q$-scheme Y, and an $\mathbb{F}_q$-algebra R, and consider a point $(P_1, \dots, P_d) \in Y^d(R)$. Its Frobenius twist is $(P_1^{(1)}, \dots, P_d^{(1)})$.*

Proof. The i-th projection $\pi_i : Y^d \to Y$ is such that $\pi_i \circ (P_1, \dots, P_d) = P_i$. By Remark 3.3.2:

$$\pi_i \circ \left((P_1, \dots, P_d)^{(1)}\right) = \pi_i \circ (P_1, \dots, P_d) \circ \mathrm{Frob}_R = P_i \circ \mathrm{Frob}_R = P_i^{(1)}. \qquad \square$$

Remark 3.3.5. The analogous statement, with the same proof, is true for any product of $\mathbb{F}_q$-schemes.

Let L/K_∞ be a finite field extension with residue field $\mathbb{F}_L \subseteq \mathcal{O}_L$ and Y a proper $\mathcal{O}_L$-scheme. Recall the notation red_L from Definition 3.1.4.

Proposition 3.3.6. *Fix a point $P \in Y(L)$, and set k_L such that $\#\mathbb{F}_L = q^{k_L}$. The sequence $(P^{(mk_L+r)})_m$ converges to $\mathrm{red}_L(P)^{(r)}$ in $Y(L)$.*

Proof. Since $\mathrm{Spec}(\mathcal{O}_L)$ only has one closed point, we can choose an open affine subscheme $U \subseteq Y$ with $B := \mathcal{O}_Y(U)$ such that $P \in U(\mathcal{O}_L)$: P corresponds to a map of $\mathcal{O}_L$-algebras $\chi_P : B \to \mathcal{O}_L$; its reduction modulo $\mathfrak{m}_L$, composed with the immersion $\mathbb{F}_L \hookrightarrow \mathcal{O}_L$, is equal to the morphism $\chi_{\mathrm{red}_L(P)} : B \to \mathcal{O}_L$ corresponding to $\mathrm{red}_L(P)$ by Definition 3.1.4. For all i, $P^{(i)}$ corresponds to the map $(\cdot)^{q^i} \circ \chi_P$, which modulo $\mathfrak{m}_L^{q^i}$ is the same as $\chi_{\mathrm{red}_L(P)^{(i)}}$, hence the projections of $P^{(i)}$ and $\mathrm{red}_L(P)^{(i)}$ onto $Y(\mathcal{O}_L/\mathfrak{m}_L^{q^i})$ coincide. Since $\mathrm{red}_L(P)^{(mk_L+i)} = \mathrm{red}_L(P)^{(i)}$ for all $m \geq 0$, this proves the convergence. $\square$

Remark 3.3.7. For any effective divisor D of the curve X_L over $\mathrm{Spec}(L)$, we can define its twist $D^{(1)}$ as the pullback along F_L^X. Obviously, if $D = \sum P_i$ with $P_i \in X_L(L_i)$, $D^{(1)} = \sum P_i^{(1)}$.

Definition 3.3.8. Let h be a non-constant rational function on X_L, i.e. a non-constant morphism of L-schemes $X_L \to \mathbb{P}^1_L$. We define the Frobenius twist $h^{(1)} := h \circ F_L^X$.

Remark 3.3.9. The field of rational functions of X_L is $\mathrm{Frac}(L \otimes A)$, and if $h = \sum_i l_i \otimes a_i$ in $L \otimes A$ is non-constant, $h^{(1)} = \sum_i l_i^q \otimes a_i$. In particular, we can naturally extend the Frobenius twist to the constant rational functions $L \otimes \mathbb{F}_q \subseteq L \otimes A$ as the elevation to the q-th power.

We show that the Frobenius twists of divisors and rational functions are compatible.

Lemma 3.3.10. *Let h be a nonzero rational function on X_L, and call $\mathrm{Div}(h)$ its divisor. Then, $\mathrm{Div}(h^{(1)})$ is equal to $(\mathrm{Div}(h))^{(1)}$.*

Proof. If h is a constant function, both sides are the empty divisor. If h is non-constant, for any closed point $P \in \mathbb{P}^1_L$, $(h^{(1)})^*(P) = (F_L^X)^* \circ h^*(P)$; setting $P = [0 : 1]$ and $P = [1 : 0]$, since the Frobenius twist on the divisors is induced by the pullback via F_L^X, we get our thesis. □

3.3.2 Rational functions on X_{K_∞} as Laurent series

Let L/K_∞ be a finite field extension with a uniformizer $u \in \mathcal{O}_L$, call $\mathbb{F}_L \subseteq \mathcal{O}_L$ the residue field of L, and k_L the integer such that $\#\mathbb{F}_L = q^{k_L}$. Call $K' := \mathbb{F}_L \otimes K$, i.e. the fraction field of $X_{\mathbb{F}_L}$, and $A' := \mathbb{F}_L \otimes A$.

Remark 3.3.11. Since the field of rational functions on X_L is the fraction field of $\mathcal{O}_L \otimes K$, which has as a maximal ideal $\mathfrak{m}_L \otimes K = (u \otimes 1)\mathcal{O}_L \otimes K$, we can endow it with the $\mathfrak{m}_L \otimes K$-adic metric.

Lemma 3.3.12. *The field of rational functions on X_L can be naturally immersed in $K'((u))$, and this immersion is a completion with respect to the $\mathfrak{m}_L \otimes K$-adic metric. Moreover, it induces an isomorphism between the completion $L\hat{\otimes}A$ of $L \otimes A = H^0(\mathcal{O}_{X_L}, (X \setminus \infty)_L)$ and $A'[[u]][u^{-1}]$.*

Proof. The natural isomorphisms $(\mathcal{O}_L \otimes K/\mathfrak{m}_L^k \otimes K \cong K'[u]/u^n)_{n \geq 1}$ pass to the limit and to fraction fields, giving a natural isometry between the completion of the field of rational functions on X_L and $K'((u))$.

The inclusion $L \otimes A \subseteq A'[[u]][u^{-1}]$ is obvious, and by the previous reasoning it is an isometry with respect to the natural metric of $L \otimes A$; on the other hand each element in $A'[[u]][u^{-1}]$ is the limit of its truncated expansions, which are in $L \otimes A$. □

Lemma 3.3.13. *For all positive integers d, the induced inclusion of $H^0(X_L, d\infty)$, endowed with its natural metric of finite L-vector space, into $K'((u))$ is a closed immersion.*

Proof. The restriction of the $\mathfrak{m}_L \otimes K$-adic metric of the field of rational functions on X_L to the space $H^0(X_L, d\infty)$, which is isomorphic to $L \otimes A(\leq d)$, is the natural metric of a finite L-vector space. By Lemma 3.3.12, the inclusion into $K'((u))$ is an isometry, hence a closed immersion. □

Remark 3.3.14. For any rational function h on X_L, if we write $h = \sum_{j \geq m} c_j u^j \in K'((u))$, with $c_j \in K'$ for all j, then $h^{(1)} = \sum_{j \geq m} c_j^{(1)} u^{qj}$.

To better understand the usefulness of $K'((u))$, let's state a couple of propositions. First, we prove a very natural result, analogous to Lemma 3.3.10 but with the reduction instead of the twist.

Definition 3.3.15. For all nonzero rational functions h on X_L, write $h = \sum_{j\geq m} c_j u^j \in K'((u))$ with $c_j \in K'$ for all j and $c_m \neq 0$, and set $\mathrm{red}_u(h) := c_m$.

Proposition 3.3.16. *For all nonzero rational functions h on X_L, $\mathrm{Div}(\mathrm{red}_u(h)) = \mathrm{red}_L(\mathrm{Div}(h))$, where both are $\mathbb{F}_L$-rational divisors of $X_{\mathbb{F}_L}$.*

Proof. Since for any nonzero rational function h on X_L there is a positive integer d and h_+, h_- in $\mathcal{O}_L \otimes_{\mathbb{F}_L} A'(\leq d)$ such that $h = \frac{h_+}{h_-}$, we can assume $h \in \mathcal{O}_L \otimes_{\mathbb{F}_L} A'(\leq d)$. Up to a factor in $L^\times$, we can also assume $h = \sum_{i\geq 0} c_i u^i \in K'[[u]]$ with $c_0 \in A'(\leq d) \setminus \{0\}$. By Remark 3.3.14, the sequence $(h^{(mk_L)})_m$ is equal to $(\sum_{j\geq 0} c_j u^{jq^{mk_L}})_m$, hence it converges to c_0 in $K'[[u]]$; by Lemma 3.3.13 this convergence lifts to $L \otimes_{\mathbb{F}_L} A'(\leq d)$. The sequence of divisors $(\mathrm{Div}(h^{(mf_L)}) + d\infty)_m$, by Proposition 3.2.9, converges to $\mathrm{Div}(c_0) + d\infty$ in $X^{[d]}(L)$; on the other hand, by Proposition 3.3.6, it converges to $\mathrm{red}_L(\mathrm{Div}(h)) + d\infty$, hence we have the desired equality. □

We prove now that the immersion of the field of rational functions on X_L in $K'((u))$ behaves reasonably well with evaluations.

Proposition 3.3.17. *Fix $h \in H^0(X_L, d\infty)$, and expand $h = \sum_i h_{(i)} u^i$ as an element of $K'((u))$; fix $P \in X_{\mathbb{F}_L}(\bar{L}) \setminus \{\infty\}$, corresponding to a $\mathbb{F}_L$-linear homomorphism $\chi_P : A' \to \bar{L}$. Then, $h_{(i)} \in A'$ for all i and $h(P) = \sum_i \chi_P(h_{(i)}) u^i$.*

Proof. We can write $h = \sum_j \gamma_j a_j$, with $\gamma_j \in L = \mathbb{F}_L((u))$ and $a_j \in A'(\leq d)$, hence $h_{(i)} \in A'(\leq d)$ for all i. For all integers m define $\gamma_{j,m}$ as the truncation of $\gamma_j \in \mathbb{F}_L((u))$ at the degree m, and define $h_m := \sum_j \gamma_{j,m} a_j \in K'[u^{\pm 1}]$, so that $h_m = \sum_{i\leq m} h_{(i)} u^i$. We have the equalities:

$$h_m(P) = \chi_P\left(\sum_j \gamma_{j,m} a_j\right) = \sum_j \gamma_{j,m} \chi_P(a_j);$$

$$h_m(P) = \chi_P\left(\sum_{i\leq m} h_{(i)} u^i\right) = \sum_{i\leq m} \chi_P(h_{(i)}) u^i;$$

where we used that both summations are finite. Since the sequence $(\gamma_{j,m})_m$ converges to γ_j in $\mathbb{F}_L((u))$ for all j, the first equation tells us that the sequence $(h_m(P))_m$ converges to $h(P)$. From the second equation we deduce that the series $\sum_i \chi_P(h_{(i)}) u^i$ also converges, and is equal to $h(P)$. □

Proposition 3.3.18. *Let $h = \sum_i h_{(i)} u^i \in A'[[u]][u^{-1}]$ be a rational function on X_L, and fix $P \in X_{\mathbb{F}_L}(L)$ such that $\mathrm{red}_L(P) \neq \infty$, corresponding to a $\mathbb{F}_L$-linear homomorphism $\chi_P : A' \to \mathcal{O}_L$. Then P is not a pole of h, and $h(P) = \sum_i \chi_P(h_{(i)}) u^i$.*

Proof. For $N \gg 0$, the space $H^0(X_L, N\infty - \mathrm{Div}_-(h) - P)$ is strictly included in $H^0(X_L, N\infty - \mathrm{Div}_-(h))$; we can fix h_- in their difference, and set $h_+ := hh_-$; by definition, $h_+, h_- \in L \otimes A$.

If we write $h_+ = \sum_i h_{+,(i)} u^i$ and $h_- = \sum_i h_{-,(i)} u^i$, we have for all integers k the equation $h_{+,(k)} = \sum_{i+j=k} h_{(i)} h_{-,(j)}$, which commutes with evaluation, being a finite sum. Since χ_P has image in $\mathcal{O}_L$, the series $\sum_i \chi_P(h_{(i)}) u^i$ converges, hence by Proposition 3.3.17 we get the following equation in $\mathcal{O}_L$:

$$\begin{aligned} h_+(P) &= \sum_k \chi_P(h_{+,(k)}) u^k = \sum_k \sum_{i+j=k} \chi_P(h_{(i)}) \chi_P(h_{-,(j)}) u^k \\ &= \left(\sum_i \chi_P(h_{(i)}) u^i\right)\left(\sum_j \chi_P(h_{-,(j)}) u^j\right) = \left(\sum_i \chi_P(h_{(i)}) u^i\right) h_-(P). \end{aligned}$$

Since $h_- \in H^0(X_L, N\infty - \mathrm{Div}_-(h)) \setminus H^0(X_L, N\infty - \mathrm{Div}_-(h) - P)$, if P is a pole of h, then P is a zero of h_- of the same order, and $h_+(P) \neq 0$, hence we reach a contradiction by the previous equation. Since P is not a pole of h, then $h_-(P) \neq 0$, and since $h_+(P) = h(P)h_-(P)$ we get:

$$h(P) = \frac{h_+(P)}{h_-(P)} = \sum_i \chi_P(h_{(i)})u^i. \qquad \square$$

Proposition 3.3.19. *Let $h = \sum_i h_{(i)}u^i \in K'((u))$ be a rational function on X_L. Then, h is in $A'[[u]][u^{-1}]$ if and only if all its poles reduce to ∞.*

Proof. Suppose $h \in A'[[u]][u^{-1}]$ and take a pole $P \in X_L(E)$ of h, where E/L is a finite field extension. Define $A'' := \mathbb{F}_E \otimes A$ and $K'' := \mathbb{F}_E \otimes K$, where $\mathbb{F}_E$ is the residue field of E, and fix a uniformizer v of $\mathcal{O}_E$. The natural immersion $K'((u)) \subseteq K''((v))$ sends h into $A''[[v]][v^{-1}]$; applying Proposition 3.3.18 to the function h, defined over the field E, $\mathrm{red}_E(P) = \infty$.

If vice versa $h \notin A'[[u]][u^{-1}]$, call m the least integer such that $h_{(m)} \notin A'$ and set $h' := \sum_{i<m} h_{(i)}u^i$. By Lemma 3.3.16:

$$\mathrm{Div}(h_{(m)}) = \mathrm{Div}(\mathrm{red}_u(h - h')) = \mathrm{red}_L(\mathrm{Div}(h - h')),$$

therefore, since $h_{(m)} \notin A'$, $h - h'$ has a pole at a point P which does not reduce to ∞; on the other hand, since $h' \in A'[[u]][u^{-1}]$, h' does not have a pole at P, hence P is a pole of $h = (h - h') + h'$. $\square$

Corollary 3.3.20. *Let $h = \sum_i h_{(i)}u^i \in A'[[u]][u^{-1}]$ be a nonzero rational function on X_L, and suppose that the coefficients $(h_{(i)})_i$ are all contained in some maximal ideal $P < A'$. Then P, as a closed point of X_L, is a zero of h.*

Proof. Take a point $Q \in X_L(\bar{L})$ with support at P. By Proposition 3.3.18, we get the identity $h(Q) = \sum_i h_{(i)}(Q)u^i = 0$, hence P is a zero of h. $\square$

3.3.3 Notable divisors and convergence results

As foreshadowed by Lemma 3.3.10, in this subsection we explore the relation between Frobenius twists, divisors, and the natural compact topology.

We also present an alternative construction of the *Drinfeld divisors* (see [Tha93]) from a topological point of view.

Lemma 3.3.21 (Drinfeld's vanishing lemma)**.** *Let E/K be a field extension, W a point in $X^{[d]}(E)$ for some $d \leq g$, $P, Q \in X(E)$. Suppose that $[W - W^{(m)}] = [P - Q]$, where $P \neq Q^{(sm)}$ for $0 \leq s + d \leq 2g$; then $d = g$ and $h^0(W) = 1$.*

Proof. Call $W_0 := W$ and set $W_{i+1} = W_i + Q^{(im)}$ for all $i \in \mathbb{Z}$. Note that, since $\deg(W_k) = d + k$, $h^0(W_{-d-1}) = 0$ and $h^0(W_{2g-d-1}) = g$. For all i, we have the inequalities $h^0(W_i) \leq h^0(W_{i+1}) \leq h^0(W_i) + 1$, so there is a least integer $k \in [-d, 0]$ such that $h^0(W_k) = 1$.

Let's prove that for all $i \in [-d, 2g - d - 1[$, if $h^0(W_i) \geq 1$, then $h^0(W_{i+1}) = h^0(W_i) + 1$. We have two relations:

$$\begin{aligned} W_{i+1} &= (W + Q + \cdots + Q^{((i-1)m)}) + Q^{(im)} = W_i + Q^{(im)}, \\ W_{i+1} &= (W + Q^{(m)} + \cdots + Q^{(im)}) + Q = (W_i^{(m)} - W^{(m)} + W) + Q \sim W_i^{(m)} + P; \end{aligned}$$

they imply that $H^0(X_E, W_i^{(m)}) \subseteq H^0(X_E, W_{i+1})$ and $H^0(X_E, W_i) \subseteq H^0(X_E, W_{i+1})$. To prove that those inclusions are strict, we need that $H^0(X_E, W_i) \neq H^0(X_E, W_i^{(m)})$ as subspaces of $H^0(X_E, W_{i+1})$;

they have the same dimension because the m-th Frobenius twist induces an isomorphism between the two vector spaces, so we just need $W_i \not\sim W_i^{(m)}$, but:

$$W_i \not\sim W_i^{(m)} \Leftrightarrow W - W^{(m)} \not\sim Q^{(im)} - Q \Leftrightarrow P \not\sim Q^{(im)} \Leftrightarrow P \neq Q^{(im)},$$

which is implied by our hypothesis. In particular, since W_{2g-d-1} has degree $2g-1$, we get that

$$g = h^0(W_{2g-d-1}) = h^0(W_k) + 2g - d - 1 - k = 2g - d - k,$$

therefore $g = d + k$; but $k \leq 0$ and $d \leq g$ implies $d = g$ and $k = 0$, therefore $h^0(W) = 1$. □

The previous lemma ensures that if such a divisor W exists, it has no other effective divisors in its same equivalence class. On the other hand, the existence of such W in some particular cases is ensured by the following results.

As usual, let L be a finite field extension of K_∞, with residue field $\mathbb{F}_L \subseteq \mathcal{O}_L$ and $q^{k_L} := \#\mathbb{F}_L$

Lemma 3.3.22. *Call $\mathcal{A}_0(L)$ the kernel of $\mathrm{red}_L : \mathcal{A}(L) \to \mathcal{A}(\mathbb{F}_L)$ (which is a continuous homomorphism). The map $\mathcal{A}(L) \to \mathcal{A}_0(L) \times \mathcal{A}(\mathbb{F}_L)$ sending a point D to the couple $(D - D^{(k_L)}, \mathrm{red}_L(D))$ is an isomorphism of topological groups.*

Proof. The map is obviously a continuous group homomorphism. Since domain and codomain are both compact and Hausdorff, it's sufficient to prove bijectivity.

On one hand, to prove injectivity, if we suppose $D - D^{(k_L)} = 0$ we have $D \in \mathcal{A}(\mathbb{F}_L)$, so if $\mathrm{red}_L(D) = 0$ we can deduce that $D = 0$.

On the other hand, to prove surjectivity, we fix $(D_0, \tilde{D}) \in \mathcal{A}_0(L) \times \mathcal{A}(\mathbb{F}_L)$ and show that they are the image of some $D \in \mathcal{A}(L)$. By Proposition 3.3.6 we have that the sequence $(D_0^{(ik_L)})_i$ converges to $\mathrm{red}_L(D_0) = 0$, hence by Corollary 3.1.11 the series $\tilde{D} + \sum_{i \geq 0} D_0^{(ik_L)}$ converges to some point $D \in \mathcal{A}(L)$. Since the Frobenius twist and the reduction red_L are continuous endomorphisms of $\mathcal{A}(L)$, we get the following equations:

$$D - D^{(k_L)} = \tilde{D} + \sum_{i \geq 0} D_0^{(ik_L)} - \tilde{D}^{(k_L)} - \sum_{i \geq 1} D_0^{(ik_L)} = D_0; \qquad \mathrm{red}_L(D) = \mathrm{red}_L(\tilde{D}) = \tilde{D},$$

hence the image of D is $(D_0, \tilde{D})$. □

From now on, given an effective divisor $W \in X^{[d]}(\overline{K_\infty})$, we denote by $J(W)$ its image via the morphism $J^d : X^{[d]} \to \mathcal{A}$, i.e. the equivalence class $[W - d\infty]$ in the Jacobian (so that, for any pair of effective divisors W and W' of arbitrary degree, $J(W + W') = J(W) + J(W')$).

The following proposition holds (cf. [Tha93][Cor. 0.3.3, Lemma 1.1]).

Proposition 3.3.23. *Fix a point $D \in \mathcal{A}(\mathbb{F}_q)$, and let $P, Q \in X(K_\infty)$ such that $\mathrm{red}_{K_\infty}(P)$ is equal to $\mathrm{red}_{K_\infty}(Q)$, with $P \neq Q^{(s)}$ for $|s| < 2g$.*

Then, there is a unique effective divisor W such that: $[W - W^{(1)}] = [P - Q]$, the reduction of $J(W)$ is D, and $\deg(W) \leq g$. Moreover, in retrospect, $W \in X^{[g]}(K_\infty)$ and, if R is a point in the support of W, $R \notin X(\overline{\mathbb{F}_q})$.

Proof. By Lemma 3.3.22 there is an element $D' \in \mathcal{A}(K_\infty)$ such that $D' - D'^{(1)} = [P - Q]$ and $\mathrm{red}_{K_\infty}(D') = D$. Since the morphism J^g is surjective, there is a divisor $W \in X^{[g]}(\overline{K_\infty})$ such that $J(W) = D'$. By Drinfeld's vanishing lemma, there is only one divisor of degree $\leq g$ with the requested properties, hence $h^0(W) = 1$; by Corollary 3.2.8, W is K_∞-rational.

Now, call $W' \leq W$ the maximal $\overline{\mathbb{F}_q}$-rational effective divisor ($W' \in X^{[d]}(\overline{\mathbb{F}_q})$), and call G the group of K_∞-linear field automorphisms of $\overline{K_\infty}$, which acts naturally on $X(\overline{K_\infty})$. Since $W \in X^{[g]}(K_\infty)$, it is fixed by the induced action of G; moreover, this action sends $X(\overline{\mathbb{F}_q})$ to itself, hence $W' \leq W$ is also fixed by G: since W' is both K_∞-rational and $\overline{\mathbb{F}_q}$-rational, $W' \in X^{[d]}(\mathbb{F}_q)$. We have:

$$(W - W') - (W - W')^{(1)} = (W - W^{(1)}) + (W' - W'^{(1)}) = W - W^{(1)} \sim P - Q,$$

but $\deg(W - W') = g$ from Drinfeld's vanishing lemma, hence $d = \deg(W') = 0$. □

Recall the notation of $I, \bar{I}, \Xi$ from the start of this section.

Lemma 3.3.24. *We have the identity* $\mathrm{red}_{K_\infty}(\Xi) = \infty$ *in* $X(K_\infty)$.

Proof. Since the image of the canonical inclusion $A \hookrightarrow K_\infty$ is not contained in $\mathcal{O}_{K_\infty}$, the morphism $\Xi : \mathrm{Spec}(K_\infty) \to X \setminus \{\infty\}$ does not factor through $\mathrm{Spec}(\mathcal{O}_{K_\infty})$, which means that $\mathrm{red}_{K_\infty}(\Xi) \notin X(\mathbb{F}_q) \setminus \{\infty\}$, so $\mathrm{red}_{K_\infty}(\Xi) = \infty$. □

Next, we construct some notable divisors.

Proposition 3.3.25. *The following effective divisors of X_{K_∞} exist and are unique:*

- *a divisor $V_{\bar{I}}$ of degree $\leq g$, such that* $\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = J(I)$ *and* $V_{\bar{I}} - V_{\bar{I}}^{(1)} \sim \Xi - \infty$;
- *for $m \geq 1$, a divisor $V_{\bar{I},m}$ of degree $\leq g$, such that* $\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = J(I)$ *and* $V_{\bar{I},m} - V_{\bar{I},m}^{(1)} \sim \Xi^{(1)} - \Xi^{(m+1)}$;
- *a divisor $V_{\bar{I},*}$ of degree $\leq g$ such that $J(V_{\bar{I},*}) + J(V_{\bar{I}}) = 0$;*
- *for $m \gg 0$, a divisor $V_{\bar{I},*,m}$ of degree $\leq g$ such that $J(V_{\bar{I},*,m}) + J(V_{\bar{I},m}) = 0$.*

Moreover, in retrospect, their degree is exactly g.

Proof. Let's first note that the divisors, if they exist, are well defined: since for all $a, b \in A \setminus \{0\}$ $J(aI) = J(bI)$, the properties of the divisors we want to construct only depend on the ideal class $\bar{I} \in Cl(A)$ of I.

Since $\mathrm{red}_{K_\infty}(\Xi) = \infty$ by Lemma 3.3.24, we can apply Proposition 3.3.23 to $P = \Xi$ and $Q = \infty$ (resp. $P = \Xi^{(1)}$ and $Q = \Xi^{(m+1)}$ for $m \gg 0$), so the divisor $V_{\bar{I}}$ (resp. $V_{\bar{I},m}$) exists, is unique, and is contained in $X^{[g]}(K_\infty)$.

Since $J^g(\overline{K_\infty}) : X^{[g]}(\overline{K_\infty}) \to \mathcal{A}(\overline{K_\infty})$ is surjective, there is at least one effective divisor $V_{\bar{I},*}$ of degree at most g such that $J(V_{\bar{I},*}) = -J(V_{\bar{I}})$. It has the following properties:

$$[V_{\bar{I},*} - V_{\bar{I},*}^{(1)}] = [V_{\bar{I}}^{(1)} - V_{\bar{I}}] = [\infty - \Xi]; \qquad \mathrm{red}_{K_\infty}(J(V_{\bar{I},*})) = -\mathrm{red}_{K_\infty}(J(V_{\bar{I}})) = -J(I).$$

By Proposition 3.3.23 applied to $P = \infty$ and $Q = \Xi$, $V_{\bar{I},*}$ is unique, K_∞-rational, and of degree g.

Similarly, the existence and uniqueness of $V_{\bar{I},*,m}$ for $m \gg 0$ are ensured by the following properties: $V_{\bar{I},*,m} - V_{\bar{I},*,m}^{(1)} \sim \Xi^{(m+1)} - \Xi^{(1)}$, and $\mathrm{red}_{K_\infty}(J(V_{\bar{I},*,m})) = -J(I)$. □

Remark 3.3.26. If we fix an inclusion $H \subseteq K_\infty$, where H is the Hilbert class field of K, the *Drinfeld divisors* $V_{\bar{I}}$ are actually H-rational, and the natural action of $\mathrm{Gal}(H/K)$ on the set $\{V_{\bar{I}}\}_{\bar{I}\in Cl(A)}$ is free and transitive (see [Hay79][Prop. 3.2, Thm. 8.5]). Call $\bar{I}^\sigma \in Cl(A)$ the element such that $V_{\bar{I}^\sigma} = V_{\bar{I}}^\sigma$. Since this action commutes with morphisms of schemes, for all $\sigma \in \mathrm{Gal}(H/K)$, for all $\bar{I} \in Cl(A)$, we have that

$$[V_{\bar{I},*}^\sigma - g\infty] = [V_{\bar{I},*} - g\infty]^\sigma = [g\infty - V_{\bar{I}}]^\sigma = [g\infty - V_{\bar{I}}^\sigma] = [g\infty - V_{\bar{I}^\sigma}] = [V_{\bar{I}^\sigma,*} - g\infty];$$

hence $V_{\bar{I},*}^\sigma = V_{\bar{I}^\sigma,*}$ by Proposition 3.3.25 because of uniqueness.

Finally, we state the main result of this subsection, which is central to the proof of the main theorems.

Proposition 3.3.27. *The sequences $(V_{\bar{I},m})_m$ and $(V_{\bar{I},*,m})_m$ converge respectively to the divisors $V_{\bar{I}}^{(1)}$ and $V_{\bar{I},*}^{(1)}$ in $X^{[g]}(K_\infty)$.*

Proof. Define $U := \{D \in X^{[g]}(K_\infty) | h^0(D) = 1\}$, so that the restriction $J^g(K_\infty)|_U$ induces a bijection of U with its image in $\mathcal{A}(K_\infty)$; by definition, U is the preimage of its image, hence by Lemma 3.1.7 the restriction $J^g(K_\infty)|_U$ is a homeomorphism. By Proposition 3.3.25, for $m \gg 0$, $h^0(V_{\bar{I},m}) = h^0(V_{\bar{I}}^{(1)}) = 1$, so $V_{\bar{I},m}, V_{\bar{I}}^{(1)} \in U$, and it suffices to prove the convergence of their images in $\mathcal{A}(K_\infty)$.

If we identify $\mathcal{A}(K_\infty)$ and $\mathcal{A}(\mathbb{F}_q) \times \mathcal{A}_0(K_\infty)$ by Lemma 3.3.22, we have:

$$\begin{aligned}\lim_m V_{\bar{I},m} &= \lim_m \left(\mathrm{red}_{K_\infty}(J^g(V_{\bar{I},m})), [V_{\bar{I},m} - V_{\bar{I},m}^{(1)}]\right) = \lim_m \left(J(I), [\Xi^{(1)} - \Xi^{(m+1)}]\right) \\ &= \left(J(I), [\Xi^{(1)} - \infty]\right) = \left(\mathrm{red}_{K_\infty}(J^g(V_{\bar{I}}^{(1)})), [V_{\bar{I}}^{(1)} - V_{\bar{I}}^{(2)}]\right) = V_{\bar{I}}^{(1)},\end{aligned}$$

where we used that $\lim_m \Xi^{(m)} = \infty$ in $X(K_\infty)$ by Lemma 3.3.24 and Proposition 3.3.6. Similarly, for the other statement, it suffices to prove that the sequence $(J(V_{\bar{I},m,*}))_m$ converges to $J(V_{\bar{I},*}^{(1)})$, which is obvious because $J(V_{\bar{I},*}^{(1)}) = -J(V_{\bar{I}}^{(1)})$ and $J(V_{\bar{I},m,*}) = -J(V_{\bar{I},m})$ for all $m \gg 0$. □

Chapter 4

Pellarin-type identities in Drinfeld A-modules of rank 1

We adopt the notation of the previous chapter, and consider the degree map $\deg : A \to \mathbb{Z}$ and the norm $\|\cdot\| : \mathbb{C}_\infty \to \mathbb{R}_{>0}$ as defined in Section 1.3. We also assume $\infty \in X$ to be $\mathbb{F}_q$-rational.

Pellarin's zeta function, first introduced in [Pel12] in the case $A = \mathbb{F}_q[\theta]$, is defined as the following series:

$$\zeta_A := -\sum_{a \in A} a^{-1} \otimes a \in \mathbb{C}_\infty \hat{\otimes} A.$$

As we explained in the introduction, Pellarin proved in his original article that the product of ζ_A with the Anderson–Thakur special function is a rational function on $\mathbb{P}^1_{\mathbb{C}_\infty}$, and this rationality was also proved by Green and Papanikolas when X is an elliptic curve in [GP18]. Theorem 4.3.32 generalizes this result to curves of arbitrary genus.

Let's give an intuitive explanation on why we should expect a similar theorem to hold. On one hand, given a Drinfeld module $(\mathbb{G}_a, \phi)$, the defining property of a special function $\omega \in \mathbb{C}_\infty \hat{\otimes} A$ is that

$$\omega^{(1)} = f\omega,$$

where f is the shtuka function relative to ϕ, with divisor $V^{(1)} - V + \Xi - \infty$ (recall Proposition 2.3.1 and Remark 2.3.2). This property suggests that we could define a special function ω as an infinite product similar to

$$\omega = \left(\prod_{i \geq 0} f^{(i)} \right)^{-1};$$

with some adjustments, we prove a similar formula in Theorem 4.2.6 and Theorem 4.3.29. Using this formula we can compute a formal divisor of ω outside the point at ∞ (which we temporarily denote by Div') in the following way:

$$\mathrm{Div}'(\omega) = -\lim_k \mathrm{Div}' \left(\prod_{i \geq 0}^{k-1} f^{(i)} \right) = -\lim_k (V^{(k)} - V + \Xi + \cdots + \Xi^{(k-1)}) = V - \lim_k V^{(k)} - \Xi - \Xi^{(1)} - \ldots.$$

On the other hand, Pellarin's zeta $\zeta_A \in \mathbb{C}_\infty \hat{\otimes} A$ can be interpreted as a function from $X(\mathbb{C}_\infty) \setminus \{\infty\}$ to $\mathbb{C}_\infty$, and—as proven by Chung, Ngo Dac and Pellarin in [CNP23]—is 0 when evaluated at $\Xi^{(n)}$ for all $n \geq 0$. Therefore, if we could attach to ζ_A a formal divisor outside the point at ∞, it would be:

$$\mathrm{Div}'(\zeta_A) = W + \Xi^{(1)} + \cdots + \Xi^{(n)} + \ldots$$

for some unknown formal divisor W, which represents the set of "nontrivial" zeros of ζ_A. Multiplying with the special function ω, we obtain:

$$\mathrm{Div}'(\zeta_A\omega) = W + V - \lim_k V^{(k)}.$$

This suggests that to prove the rationality of the product $\zeta_A\omega$ we need to prove that W is a proper divisor, or in other words that the set of "nontrivial" zeros of ζ_A is finite.

In this chapter we avoid the use of "formal divisors" by instead using well-behaved rational approximations of the functions ω and ζ_A, but the reasoning we have just outlined is the guiding principle behind the proofs of the main results.

4.1 Pellarin zetas

Throughout this and the next sections, we fix a uniformizer $u \in K$ of K_∞ and a nonzero ideal $I < A$. As in the previous section, we also call I the corresponding closed subscheme of $X \setminus \{\infty\}$, and d its degree. For any rational function h on $X_{\mathbb{C}_\infty}$, we denote by $\mathrm{sgn}(h)$ the sign of h at ∞ with respect to $1 \otimes u$—which is a rational function on $X_{\mathbb{C}_\infty}$ with a zero of degree 1 at ∞—so that for all $a \in A \setminus \{0\}$ the sign $\mathrm{sgn}(1 \otimes a)$ is equal to $\mathrm{red}_u(a \otimes 1)$.

The following definition is a generalization of the zeta functions à la Pellarin introduced in [Pel12].

Definition 4.1.1. The (partial) *Pellarin zeta* relative to I is defined as the series:

$$\zeta_I := -\sum_{a \in I\setminus\{0\}} a^{-1} \otimes a \in K_\infty \hat{\otimes} A.$$

In this section, we first define the rational approximations $\{\zeta_{I,m}\}$ of ζ_I and compute their divisors, following the proof of Chung, Ngo Dac and Pellarin in the case $I = A$ (see [CNP23][Lemma 2.1]). Afterwards, we use Proposition 3.2.9 to prove a functional identity regarding ζ_I in the shape of an infinite product, i.e. Theorem 4.1.9.

A strengthening of this result (Theorem 4.3.28, stated in the introduction) is proven at the end of Section 4.3.

4.1.1 The approximations of ζ_I and their divisors

For $m \in \mathbb{N}$, call j_m the least integer such that $\dim_{\mathbb{F}_q}(I(\leq j_m)) = h^0(j_m\infty - I) = m + 1$. We call $a_I \in I$ the nonzero element with least degree (i.e. $a_I \in I(j_0)$) and sign 1.

Remark 4.1.2. Since $\deg(j_m\infty - I) + 1 - g \leq h^0(j_m\infty - I) \leq \deg(j_m\infty - I) + 1$, we get the inequality:

$$m + d \leq j_m \leq m + g + d.$$

Moreover, for $m \gg 0$, the rightmost inequality becomes an equality.

Definition 4.1.3. We set for all $m \geq 0$:

$$\zeta_{I,m} := -\sum_{a \in I(\leq j_m)\setminus\{0\}} a^{-1} \otimes a \in A_{K_\infty}.$$

Remark 4.1.4. The sequence $\zeta_{I,m}$ converges to ζ_I in $K_\infty \hat{\otimes} A \cong A[[u]]$.

Proposition 4.1.5. *The divisor of $\zeta_{I,m}$ is $\Xi^{(1)} + \cdots + \Xi^{(m)} + I + W_m - j_m\infty$ for some effective divisor W_m with $h^0(W_m) = 1$. Moreover, for $m \gg 0$, $j_m = m + g + d$ and $W_m = V_{\bar{I},*,m}$.*

The following result is similar to a well known lemma (see [Gos98][Lemma 8.8.1]). We prove it in this stronger form because of its use in Section 4.3.

Lemma 4.1.6. *Call $S_{n,d}(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, \ldots, x_n]$ the sum of the d-th powers of all the homogeneous linear polynomials. Suppose that the coefficient of monomial $x_1^{d_1} \cdots x_n^{d_n}$ in the expansion of $S_{n,d}(x_1, \ldots, x_n)$ is nonzero: then, for all $1 \leq j \leq n$, $\sum_{i=1}^{j} d_i \geq q^j - 1$. In particular, if $d < q^n - 1$, $S_{n,d} = 0$.*

Proof. The coefficient $c_{d_1,\ldots,d_n}$ of the monomial $x_1^{d_1} \cdots x_n^{d_n}$ is:

$$\frac{d!}{d_1! \cdots d_n!} \sum_{a_1,\ldots,a_n \in \mathbb{F}_q} a_1^{d_1} \cdots a_n^{d_n} = \frac{d!}{d_1! \cdots d_n!} \prod_{i=1}^{n} \left(\sum_{a_i \in \mathbb{F}_q} a_i^{d_i} \right),$$

where by convention we set $0^0 = 1$. On one hand, if the multinomial coefficient $\frac{d!}{d_1! \cdots d_n!}$ is nonzero in $\mathbb{F}_q$, $C(d) = C(d_1) + \cdots + C(d_n)$, where we denote by $C(m)$ the sum of the digits in base q of the nonnegative integer m; in particular, for $1 \leq j \leq n$ this implies $C(d_1 + \cdots + d_j) = C(d_1) + \cdots + C(d_j)$. On the other hand, $\sum_{a_i \in \mathbb{F}_q} a_i^{d_i} \neq 0$ if and only if $d_i > 0$ and $q-1|d_i$; in particular, this implies $C(d_i) \geq q-1$ for all i.

If $c_{d_1,\ldots,d_n} \neq 0$, for $1 \leq j \leq n$ we have:

$$C\left(\sum_{i=1}^{j} d_i\right) = \sum_{i=1}^{j} C(d_i) \geq (q-1)j,$$

hence $\sum_{i=1}^{j} d_i \geq q^j - 1$. Applying this to $j = n$ we get the condition $d \geq q^n - 1$, therefore $S_{n,d} = 0$ for all $d < q^n - 1$. □

Proof of Proposition 4.1.5. Since $\zeta_{I,m}$ is sum of elements whose divisor contains I, it's obvious that $\mathrm{Div}^+(\zeta_{I,m}) \geq I$. If we fix an $\mathbb{F}_q$-basis $\{a_i\}_{i=0,\ldots,m}$ of $I(\leq j_m)$, for any positive integer k we have:

$$\zeta_{I,m}(\Xi^{(k)}) = -\sum_{a \in I(\leq j_m)} a^{q^k - 1} = -S_{m+1,q^k-1}(a_0, \ldots, a_m),$$

which by Lemma 4.1.6 is zero when $k \leq m$. Since the only poles are at ∞, and have multiplicity at most j_m, $\mathrm{Div}(\zeta_{I,m}) = \Xi^{(1)} + \cdots + \Xi^{(m)} + I + W_m - j_m\infty$ for some effective divisor W_m. To study $h^0(W_m)$, call $D_n := j_m\infty - I - \sum_{i=1}^{n} \Xi^{(i)}$ for all nonnegative integers n.

Note that, since $(j_m\infty - I)^{(1)} = j_m\infty - I$, for all $n \geq 0$:

$$\begin{aligned}
&H^0(X_{\overline{K_\infty}}, D_{n+1}) \subseteq H^0(X_{\overline{K_\infty}}, D_n), \\
&H^0(X_{\overline{K_\infty}}, D_{n+1}) \subseteq H^0(X_{\overline{K_\infty}}, D_n^{(1)}), \\
&H^0(X_{\overline{K_\infty}}, D_n) \cap H^0(X_{\overline{K_\infty}}, D_n^{(1)}) = H^0(X_{\overline{K_\infty}}, D_{n+1}).
\end{aligned}$$

Let's prove that, for all $n \geq 0$, if $h^0(D_n) \geq 1$, then $h^0(D_{n+1}) = h^0(D_n) - 1$. By contradiction, assume that the set $S := \{n \in \mathbb{Z}_{\geq 0} | h^0(D_{n+1}) = h^0(D_n) > 0\}$ is not empty. Since for $k \gg 0$ $h^0(D_k) = 0$, S admits a maximum element n; since $D_n^{(1)} > D_{n+1}$ and $D_n^{(1)} > D_{n+1}^{(1)}$, we have

$H^0(X_{\overline{K_\infty}}, D_{n+1}^{(1)}) + H^0(X_{\overline{K_\infty}}, D_{n+1}) \subseteq H^0(X_{\overline{K_\infty}}, D_n^{(1)})$; since $h^0(D_{n+1}) = h^0(D_n) = h^0(D_n^{(1)})$ we get the following identities:

$$\begin{aligned}
&H^0(X_{\overline{K_\infty}}, D_n^{(1)}) = H^0(X_{\overline{K_\infty}}, D_{n+1}) = H^0(X_{\overline{K_\infty}}, D_{n+1}^{(1)})\\
\Rightarrow &H^0(X_{\overline{K_\infty}}, D_{n+1}) = H^0(X_{\overline{K_\infty}}, D_{n+1}) \cap H^0(X_{\overline{K_\infty}}, D_{n+1}^{(1)}) = H^0(X_{\overline{K_\infty}}, D_{n+2})\\
\Rightarrow &h^0(D_{n+2}) = h^0(D_{n+1}).
\end{aligned}$$

We deduce $n+1 \in S$, which contradicts the maximality hypothesis on n, therefore $S = \emptyset$. In particular $m \notin S$, and since $h^0(W_m) \geq 1$ and $W_m \sim D_m$, we have:

$$h^0(W_m) = h^0(D_m) = h^0(D_0) - m = h^0(j_m\infty - I) - m = 1.$$

On one hand, $\deg(W_m) = \deg(j_m\infty - \Xi^{(1)} - \cdots - \Xi^{(m)} - I) = j_m - m - d$, which is $\leq g$ by Remark 4.1.2. On the other hand, by Lemma 3.3.10 and Proposition 3.3.16 we have:

$$\begin{aligned}
&0 \sim \mathrm{Div}(\zeta_{I,m}) - \mathrm{Div}(\zeta_{I,m})^{(1)} = \Xi^{(1)} - \Xi^{(m+1)} + W_m - W_m^{(1)},\\
&0 \sim \mathrm{Div}(\mathrm{red}_u(\zeta_{I,m})) \sim \mathrm{red}_{K_\infty}(\mathrm{Div}(\zeta_{I,m})) = I + \mathrm{red}_{K_\infty}(W_m) - (j_m - m)\infty;
\end{aligned}$$

so $[W_m - W_m^{(1)}] = [\Xi^{(m+1)} - \Xi^{(1)}]$, and $\mathrm{red}_{K_\infty}(J(W_m)) = -J(I)$. Therefore, for $m \gg 0$, $W_m = V_{\bar{I},*,m}$ by Proposition 3.3.25. □

4.1.2 The function ζ_I as an infinite product

Proposition 4.1.7. *There are rational functions $f'_{\bar{I},*}, f'_{\bar{I}}$ on X_{K_∞} with divisors $V_{\bar{I},*} - V_{\bar{I},*}^{(1)} + \Xi - \infty$ and $V_{\bar{I}}^{(1)} - V_{\bar{I}} + \Xi - \infty$, respectively. As elements of $K((u))$, we can assume $f'_{\bar{I},*}, f'_{\bar{I}} \in 1 + uK[[u]]$.*

Moreover, there is a rational function $\delta'_{\bar{I}}$ on X_{K_∞}, with divisor $V_{\bar{I}} + V_{\bar{I},*} - 2g\infty$, such that $\frac{\delta'^{(1)}_{\bar{I}}}{\delta'_{\bar{I}}} = \frac{f'_{\bar{I}}}{f'_{\bar{I},*}}$.

Proof. From the definition of $V_{\bar{I},*}$, the divisor $V_{\bar{I},*} - V_{\bar{I},*}^{(1)} + \Xi - \infty$ is principal, hence it is the divisor of some rational function $f'_{\bar{I},*}$ on X_{K_∞}. Moreover, by Lemma 3.3.16 we have the following identity of divisors on X:

$$\mathrm{Div}(\mathrm{red}_u(f'_{\bar{I},*})) = \mathrm{red}_{K_\infty}(\mathrm{Div}(f'_{\bar{I},*})) = \mathrm{red}_{K_\infty}(V_{\bar{I},*}) - \mathrm{red}_{K_\infty}(V_{\bar{I},*})^{(1)},$$

which is the empty divisor because, since $\mathrm{red}_{K_\infty}(V_{\bar{I},*})$ is $\mathbb{F}_q$-rational, $\mathrm{red}_{K_\infty}(V_{\bar{I},*}) = \mathrm{red}_{K_\infty}(V_{\bar{I},*})^{(1)}$. We deduce that $\mathrm{red}_u(f'_{\bar{I},*}) \in \mathbb{F}_q$, hence, up to scalar multiplication, we can assume $f'_{\bar{I},*} = 1 + O(u)$. The existence ond properties of $f'_{\bar{I}}$ can be proven in the same way.

Since $V_{\bar{I}} + V_{\bar{I},*} - 2g\infty$ is principal, it is the divisor of some rational function $\tilde{\delta}'_{\bar{I}}$ contained in $H^0(X_{K_\infty}, 2g\infty) \subseteq A[[u]][u^{-1}]$, and up to scalar multiplication we can assume $\tilde{\delta}'_{\bar{I}} = c_0 + O(u)$ for some $c_0 \in A$. We get:

$$\mathrm{Div}(\tilde{\delta}'_{\bar{I}})^{(1)} - \mathrm{Div}(\tilde{\delta}'_{\bar{I}}) = \mathrm{Div}(f'_{\bar{I}}) - \mathrm{Div}(f'_{\bar{I},*}) \Longrightarrow \frac{\tilde{\delta}'^{(1)}_{\bar{I}}}{\tilde{\delta}'_{\bar{I}}} = \lambda \frac{f'_{\bar{I}}}{f'_{\bar{I},*}}$$

for some $\lambda \in K_\infty$; moreover, by considering the expansion in $K((u))$, $\lambda = 1 + O(u)$, hence it admits a $(q-1)$-th root $\mu \in \mathcal{O}_{K_\infty}$. If we set $\delta'_{\bar{I}} := \mu^{-1}\tilde{\delta}'_{\bar{I}}$ we obtain the desired equation. □

Remark 4.1.8. The choices of $f'_{\bar{I}}, f'_{\bar{I},*}, \delta'_{\bar{I}}$ are not unique.

Recall that $a_I \in I$ is the nonzero element of least degree and sign 1.

Theorem 4.1.9 (Weak version of Thm. 4.3.28)**.** *The product* $(a_I^{-1} \otimes a_I) \prod_{i\geq 1} {f'_{\bar{I},*}}^{(i)}$ *exists in* $\mathcal{O}_{K_\infty} \hat{\otimes} K$ *and is equal to* $(\lambda \otimes 1)^{-1}\zeta_I$ *for some* $\lambda \in \mathcal{O}_{K_\infty}^\times$*. We can also write:*

$$\zeta_I = (a_I^{-1} \otimes a_I) \prod_{i\geq 0} \left((\lambda \otimes 1)^{1-q} {f'_{\bar{I},*}}^{(1)} \right)^{(i)}.$$

Proof. Let's identify $\mathcal{O}_{K_\infty} \hat{\otimes} K$ with $K[[u]]$. By Proposition 4.1.7, $f'_{\bar{I},*} = 1 + O(u)$, hence ${f'_{\bar{I},*}}^{(i)} = 1 + O(u^{q^i})$ for all $i \geq 0$, and the convergence of the infinite product is obvious. For all $m \geq 0$:

$$\mathrm{red}_u(\zeta_{I,m}) = \mathrm{red}_u \left(- \sum_{a\in I(\leq j_m)\setminus\{0\}} a^{-1} \otimes a \right) = - \sum_{\mu \in \mathbb{F}_q^\times} \mathrm{red}_u(\mu a_I)^{-1} \otimes (\mu a_I) = 1 \otimes a_I.$$

In particular, by Proposition 3.3.16, for $m \gg 0$ we have:

$$\mathrm{Div}(1 \otimes a_I) = \mathrm{Div}(\mathrm{red}_u(\zeta_{I,m})) = \mathrm{red}_{K_\infty}(\mathrm{Div}(\zeta_{I,m})) = I + \mathrm{red}_{K_\infty}(V_{\bar{I},*,m}) - (g+d)\infty;$$

since $\mathrm{red}_{K_\infty} : X^{[g]}(K_\infty) \to X^{[g]}(K_\infty)$ is a continuous map, and the sequence $(V_{\bar{I},*,m})_m$ converges to $V_{\bar{I},*}$ in $X^{[g]}(K_\infty)$ by Lemma 3.3.27, the equality passes to the limit:

$$\mathrm{red}_{K_\infty}(V_{\bar{I},*}) = \mathrm{Div}(1 \otimes a_I) + (g+d)\infty - I.$$

For $m \gg 0$ define the rational function

$$\alpha_m := {\delta'_{\bar{I}}}^{(1)} \frac{\zeta_{I,m}}{{f'_{\bar{I},*}}^{(1)} \cdots {f'_{\bar{I},*}}^{(m)}}$$

and consider its divisor:

$$\mathrm{Div}(\alpha_m) = I + V_{\bar{I},*}^{(m+1)} + V_{\bar{I},*,m} + V_{\bar{I}}^{(1)} - (3g+d)\infty \Longrightarrow \alpha_m \in H^0(X_{K_\infty}, (3g+d)\infty).$$

By Lemma 3.3.27, the sequence $(\mathrm{Div}(\alpha_m) + (3g+d)\infty)_m$ converges to

$$I + \mathrm{red}_{K_\infty}(V_{\bar{I},*}) + V_{\bar{I},*}^{(1)} + V_{\bar{I}}^{(1)} = (\mathrm{Div}(1 \otimes a_I) + (g+d)\infty) + (\mathrm{Div}({\delta'_{\bar{I}}}^{(1)}) + 2g\infty)$$

in $X^{[3g+d]}(K_\infty)$. Moreover, since the sequence $(\alpha_m)_m$ converges to ${\delta'_{\bar{I}}}^{(1)} \zeta_I \left(\prod_{i\geq 1} {f'_{\bar{I},*}}^{(i)} \right)^{-1}$ in $K((u))$, by Lemma 3.3.13 the latter is an element of $H^0(X_{K_\infty}, (3g+d)\infty)$. By Proposition 3.2.9, we have:

$$\mathrm{Div} \left({\delta'_{\bar{I}}}^{(1)} \frac{\zeta_I}{\prod_{i\geq 1} {f'_{\bar{I},*}}^{(i)}} \right) = \mathrm{Div}(\lim_m \alpha_m) = \lim_m \mathrm{Div}(\alpha_m) = \mathrm{Div}(1 \otimes a_I) + \mathrm{Div}({\delta'}_{\bar{I}}^{(1)}).$$

In particular, there is some $\lambda \in K_\infty$ (in retrospect in $\mathcal{O}_{K_\infty}$) such that:

$$\zeta_I = (\lambda \otimes 1)(a_I^{-1} \otimes a_I) \prod_{i\geq 1} {f'_{\bar{I},*}}^{(i)}.$$

As elements of $K((u))$, $\zeta_I(a_I \otimes a_I^{-1}) = 1 + O(u)$, and ${f'_{\bar{I},*}}^{(i)} = 1 + O(u)$ for all $i \geq 0$, hence $\lambda \otimes 1 = 1 + u\mathbb{F}_q[[u]] \subseteq \mathbb{F}_q((u))$. In particular, the infinite product $\prod_{i\geq 0}(\lambda^{1-q} \otimes 1)^{q^i}$ converges in $\mathbb{F}_q[[u]]$ to $\lambda \otimes 1$, so we deduce the following rearrangement:

$$\zeta_I = (a_I^{-1} \otimes a_I) \prod_{i\geq 0} \left((\lambda^{1-q} \otimes 1) {f'_{\bar{I},*}}^{(1)} \right)^{(i)}. \qquad \square$$

Definition 4.1.10. Define the functions $f_{\bar{I}}, f_{\bar{I},*}, \delta_{\bar{I}}$ respectively as the unique scalar multiples of the functions $f'_{\bar{I}}.f'_{\bar{I},*}, \delta'_{\bar{I}}$ such that $\mathrm{sgn}(f_{\bar{I}}) = \mathrm{sgn}(f_{\bar{I},*}) = \mathrm{sgn}(\delta_{\bar{I}}) = 1$.

We call $\{f_{\bar{I}}\}_{\bar{I}\in Cl(A)}$ the *shtuka functions* and $\{f_{\bar{I},*}\}_{\bar{I}\in Cl(A)}$ the *adjoint shtuka functions*.

Remark 4.1.11. We have the equality $\frac{\delta_{\bar{I}}^{(1)}}{\delta_{\bar{I}}} = \frac{f_{\bar{I}}}{f_{\bar{I},*}}$, since both sides have the same divisor and the same sign.

Example 4.1.12. Let's provide some insight on the adjoint shtuka functions in the low genus cases.

If $g = 0$, i.e. $A = \mathbb{F}_q[\theta]$, there is only one ideal class $\bar{A}$, and the divisors $V_{\bar{A}}$ and $V_{\bar{A},*}$ have degree 0: in particular, $\mathrm{Div}(f_*) = \Xi - \infty = \mathrm{Div}(f)$, hence $f_* = f = 1 \otimes \theta - \theta \otimes 1$, which is in A_{K_∞}.

If $g = 1$, i.e. X is an elliptic curve, $V_{\bar{I}}$ and $V_{\bar{I},*}$ have degree 1, and since the divisor of $\delta_{\bar{I}}$ is $V_{\bar{I}} + V_{\bar{I},*} - 2\infty \sim 0$, $V_{\bar{I},*}$ is the inverse of $V_{\bar{I}}$ with respect to the group operation on $X(K_\infty)$. Suppose that we can fix an isomorphism $A \cong \mathbb{F}_q[x,y]/(y^2 - P(x))$ with $\deg(P) = 3$. If $V_{\bar{I}} \in X(K_\infty)$ corresponds to the map $(x,y) \mapsto (a,b)$ for some $a, b \in K_\infty$, then $V_{\bar{I},*} \in X(K_\infty)$ corresponds to the map $(x,y) \mapsto (a,-b)$, and assuming $\mathrm{sgn}(x) = 1$ we can write $\delta_{\bar{I}} = 1 \otimes x - a \otimes 1 \in A_{K_\infty}$, hence:

$$f_{\bar{I},*} = f_{\bar{I}} \frac{1 \otimes x - a \otimes 1}{1 \otimes x - a^q \otimes 1}.$$

Interestingly, this function does appear in the article [GP18] by Green and Papanikolas (specifically, in Lemma 7.12, as the function "Γ"), but they don't take notice of its symmetry with the shtuka function.

Remark 4.1.13. The functions $\{f_{\bar{I}}, f_{\bar{I},*}, \delta_{\bar{I}}\}_{\bar{I}\in Cl(A)}$ all have sign equal to 1, and the positive and negative components of all their divisors are H-rational by Remark 3.3.26, so all these functions are in $\mathrm{Frac}(A_H)$. From Remark 3.3.26 we also know that, for all $\bar{I} \in Cl(A), \sigma \in G(H/K) \cong Cl(A)$:

$$\mathrm{Div}(f_{\bar{I}}^\sigma) = \mathrm{Div}(f_{\bar{I}})^\sigma = \left(V_{\bar{I}}^{(1)}\right)^\sigma - V_{\bar{I}}^\sigma + \Xi - \infty = V_{\bar{I}^\sigma}^{(1)} - V_{\bar{I}^\sigma} + \Xi - \infty = \mathrm{Div}(f_{\bar{I}^\sigma}),$$

and since both functions have sign equal to 1 we get $f_{\bar{I}}^\sigma = f_{\bar{I}^\sigma}$. Similarly, $f_{\bar{I},*}^\sigma = f_{\bar{I}^\sigma,*}$ and $\delta_{\bar{I}}^\sigma = \delta_{\bar{I}^\sigma}$.

Corollary 4.1.14. *There is $\gamma_I \in \mathbb{C}_\infty$, unique up to a factor in $\mathbb{F}_q^\times$, such that:*

$$\frac{((\gamma_I \otimes 1)\zeta_I)^{(-1)}}{(\gamma_I \otimes 1)\zeta_I} = f_{\bar{I},*}.$$

Furthermore, since $f_{\bar{I},*}, \zeta_I \in K((u))$, $\gamma_I^{q-1} \in K_\infty$.

4.2 The module of special functions

Fix an ideal $I < A$. By the *shtuka correspondence* (see [Tha93], [Gos98][Section 6.2]), we can associate a normalized Drinfeld module $(\mathbb{G}_a, \phi)$ of rank 1 to the shtuka function $f_{\bar{I}}$, with the property that, for all $a \in A$, the leading term of ϕ_a is $\operatorname{sgn}(a)\tau^{\deg(a)}$. From now on, with slight abuse of notation, we write ϕ for this Drinfeld module.

In this section, we use Theorem 4.1.9 (in its partial version) to describe somewhat explicitly the module of special functions relative to ϕ.

Set $\zeta := (\gamma_I \otimes 1)\zeta_I$, with γ_I defined as in Corollary 4.1.14, so that $\zeta^{(-1)} = f_{\bar{I},*}\zeta$.

Theorem 4.2.1 (Weak version of Theorem 4.3.32)**.** *The A-module $\operatorname{Sf}_\phi(A)$ coincides with $(\mathbb{F}_q \otimes I)\frac{\delta_{\bar{I}}}{\zeta^{(-1)}}$.*

Denote by Ω the module of Kähler differentials of A. Together with Corollary 2.2.12, this theorem implies the following.

Corollary 4.2.2. *The ideal I is isomorphic as an A-module to $\operatorname{Hom}_A(\Omega, \Lambda_\phi)$.*

Remark 4.2.3. In retrospect, we can define $f_{\bar{I}}$ as the shtuka function of the unique normalized Drinfeld module of rank 1 whose lattice is isomorphic to $I \otimes_A \Omega$.

Before the proof of Theorem 4.2.1, let's state some preliminary results.

Remark 4.2.4. By Lemma 3.3.12, we know that $K_\infty \hat{\otimes} A \cong A[[u]][u^{-1}]$. A rational function on X_{K_∞} is in $\mathbb{C}_\infty \hat{\otimes} A$ if and only if it's contained in $A[[u]][u^{-1}]$, which by Proposition 3.3.19 happens if and only if its poles all reduce to ∞.

Lemma 4.2.5. *The subset of $\mathbb{C}_\infty \hat{\otimes} K$ fixed by the Frobenius twist is $\mathbb{F}_q \otimes K$.*

Proof. Fix an $\mathbb{F}_q$-basis $\{b_i\}_i$ of K: any element $c \in \mathbb{C}_\infty \hat{\otimes} K$ can be written in a unique way as a possibly infinite sum $\sum_i a_i \otimes b_i$, with $a_i \in \mathbb{C}_\infty$ for all i. If $c = c^{(1)}$, we need to have for all i the equality $a_i^q = a_i$, hence $a_i \in \mathbb{F}_q$ for all i. □

Proof of Theorem 4.2.1. First, let's show that $(\mathbb{F}_q \otimes K)\operatorname{Sf}_\phi(A) = (\mathbb{F}_q \otimes K)\frac{\delta_{\bar{I}}}{\zeta^{(-1)}}$. Pick any $\omega \in \operatorname{Sf}_\phi(A)$; since $\omega^{(1)} = f_{\bar{I}}\omega$, $\delta_{\bar{I}}^{(1)} = \frac{f_{\bar{I}}}{f_{\bar{I},*}}\delta_{\bar{I}}$, and $\zeta = \frac{1}{f_{\bar{I},*}}\zeta^{(-1)}$, we have:

$$\left(\frac{\omega\zeta^{(-1)}}{\delta_{\bar{I}}}\right)^{(1)} = \frac{\omega^{(1)}\zeta}{\delta_{\bar{I}}^{(1)}} = \frac{(f_{\bar{I}}\omega)(f_{\bar{I},*}^{-1}\zeta^{(-1)})}{f_{\bar{I}}f_{\bar{I},*}^{-1}\delta_{\bar{I}}} = \frac{\omega\zeta^{(-1)}}{\delta_{\bar{I}}},$$

hence $\frac{\omega\zeta^{(-1)}}{\delta_{\bar{I}}} \in \mathbb{F}_q \otimes K$ by Lemma 4.2.5, or equivalently $(\mathbb{F}_q \otimes K)\omega = (\mathbb{F}_q \otimes K)\frac{\delta_{\bar{I}}}{\zeta^{(-1)}}$.

We can twist both sides of last equality and multiply them by $\gamma_I \otimes 1$: the thesis is now that $(1 \otimes \lambda)\frac{\delta_{\bar{I}}^{(1)}}{\zeta_I} \in A[[u]][u^{-1}]$ if and only if $\lambda \in I$. By Proposition 3.3.25, for all integers $m \gg 0$ there is a rational function $\delta_{\bar{I},m}$ on X_{K_∞} with divisor $V_{\bar{I},m} + V_{\bar{I},*,m} - 2g\infty$. By Proposition 3.3.27 the sequence $(V_{\bar{I},m} + V_{\bar{I},*,m})_m$ converges to $V_{\bar{I}}^{(1)} + V_{\bar{I},*}^{(1)} \in X^{[2g]}(K_\infty)$, hence by Proposition 3.2.9 we can choose each $\delta_{\bar{I},m}$ so that the sequence $(\delta_{\bar{I},m})_m$ converges to $\delta_{\bar{I}}^{(1)}$ in $K((u))$.

Suppose $\lambda \in I$, and consider the sequence $\left((1 \otimes \lambda)\frac{\delta_{\bar{I},m}}{\zeta_{I,m}}\right)_m$ in $K((u))$, whose limit is $(1 \otimes \lambda)\frac{\delta_{\bar{I}}^{(1)}}{\zeta_I}$. The divisor of the m-th element of the sequence (for $m \gg 0$) is

$$V_{\bar{I},m} - (\Xi^{(1)} + \cdots + \Xi^{(m)}) - I + (m + d - g)\infty + \operatorname{Div}(1 \otimes \lambda);$$

since $\lambda \in I$, the only poles of the function reduce to ∞, hence $(1\otimes\lambda)\frac{\delta_{\bar{I},m}}{\zeta_{I,m}} \in A[[u]][u^{-1}]$ by Proposition 3.3.19, and so does the limit.

Vice versa, suppose $(1\otimes\lambda)\frac{\delta_{\bar{I}}^{(1)}}{\zeta_I} \in A[[u]][u^{-1}]$. Since the coefficients of $(1\otimes\lambda^{-1})\zeta_I$, as a series in $K((u))$, are all contained in $\lambda^{-1}I$, $\delta_{\bar{I}}^{(1)} = \left((1\otimes\lambda)\frac{\delta_{\bar{I}}^{(1)}}{\zeta_I}\right)\left((1\otimes\lambda^{-1})\zeta_I\right)$ has all coefficients in $\lambda^{-1}I$, so the same is true for $\delta_{\bar{I}}$. If by contradiction $\lambda \notin I$, there is a nonzero prime ideal $P < A$ which divides the fractional ideal $\lambda^{-1}I$, hence all the coefficients of $\delta_{\bar{I}}$ are in $A\cap\lambda^{-1}I \subseteq P$, which by Corollary 3.3.20 means that P is a zero of $\delta_{\bar{I}}$. Since P, as a closed point of X, is $\overline{\mathbb{F}_q}$-rational and $\mathrm{Div}(\delta_{\bar{I}}) = V_{\bar{I}} + V_{\bar{I},*} - 2g\infty$, this is a contradiction because, by Proposition 3.3.23, neither $V_{\bar{I}}$ nor $V_{\bar{I},*}$ have $\overline{\mathbb{F}_q}$-rational points in their support. □

To end this section, let's include an analogous result to Theorem 4.1.9 for special functions.

Theorem 4.2.6. *There is some $\alpha \in K_\infty^\times$ such that the following element of $\mathbb{C}_\infty\hat{\otimes}K$ is well defined (up to the choice of a $(q-1)$-th root of α):*

$$\omega := (\alpha\otimes 1)^{\frac{1}{q-1}}\prod_{i\geq 0}\left(\frac{\alpha\otimes 1}{f_{\bar{I}}}\right)^{(i)}.$$

Moreover, $\omega \in (\mathbb{F}_q\otimes K)\,\mathrm{Sf}_\phi(A)$.

Proof. Fix an isomorphism $K_\infty \cong \mathbb{F}_q((u))$. By Proposition 4.1.7, we can choose some nonzero $\alpha \in \mathbb{F}_q((u))$ such that $\alpha^{-1}f_{\bar{I}} = 1 + O(u)$, hence the product $\prod_{i\geq 0}\left(\frac{\alpha\otimes 1}{f_{\bar{I}}}\right)^{(i)}$ converges in $K_\infty\hat{\otimes}K \cong K((u))$, and ω is well defined up to the choice of $\alpha^{\frac{1}{q-1}}$. We have:

$$\frac{\omega^{(1)}}{\omega} = \left((\alpha\otimes 1)^{\frac{1}{q-1}}\right)^q\prod_{i\geq 0}\left(\frac{\alpha\otimes 1}{f_{\bar{I}}}\right)^{(i+1)}\left((\alpha\otimes 1)^{\frac{1}{q-1}}\prod_{i\geq 0}\left(\frac{\alpha\otimes 1}{f_{\bar{I}}}\right)^{(i)}\right)^{-1} = f_{\bar{I}},$$

so $\omega \in (\mathbb{F}_q\otimes K)\,\mathrm{Sf}_\phi(A)$ by the same considerations expressed in the proof of Theorem 4.2.1. □

Remark 4.2.7. It's not difficult to observe that if $\beta,\gamma \in K_\infty^\times$ are such that the infinite products $\omega(\beta) := (\beta\otimes 1)^{\frac{1}{q-1}}\prod_{i\geq 0}\left(\frac{\beta\otimes 1}{f_{\bar{I}}}\right)^{(i)}$ and $\omega(\gamma) := (\gamma\otimes 1)^{\frac{1}{q-1}}\prod_{i\geq 0}\left(\frac{\gamma\otimes 1}{f_{\bar{I}}}\right)^{(i)}$ are well defined, $\omega(\beta)$ is equal to $\omega(\gamma)$ up to a factor in $\mathbb{F}_q$.

4.3 Relation between Pellarin zetas and period lattices

The aim of this section is to compute more explicitly the constant γ_I defined in Corollary 4.1.14. To do so, we first study more in depth ζ_I and its coefficients as a series in $K[[u]]$; afterwards, we draw a correspondence between the adjoint shtuka function $f_{\bar{I},*}$ introduced in Definition 4.1.10 and a certain normalized Drinfeld module ϕ of rank 1. Let's give the definition of the exponential function associated to ϕ.

Definition 4.3.1. Given a Drinfeld module $\phi : A \to \mathbb{C}_\infty[\tau]$, the exponential $\exp_\phi \in \mathbb{C}_\infty[[\tau]]$ is the unique formal series such that:

- its leading term is 1;

- for all $a \in A$, $\phi_a \exp_\phi = \exp_\phi a$.

Recall that by Proposition 2.2.3, $\exp_\phi$ converges everywhere as a function from $\mathbb{C}_\infty$ to itself, and its kernel Λ_ϕ, called period lattice, is a projective A-module of the same rank as ϕ.

Proposition (Prop. 4.3.23)**.** *The period lattice of ϕ is $\gamma_I^{-1} I \subseteq \mathbb{C}_\infty$.*

Finally, we state Theorem 4.3.32 (a strengthening of Theorem 4.2.1 which properly generalizes [GP18][Thm. 7.1]) and Theorem 4.3.28 (a strengthening of Theorem 4.1.9).

4.3.1 Evaluations of the Pellarin zetas

The aim of this subsection, expressed in the following proposition, is to show that there is a well behaved notion of evaluation for the Pellarin zeta ζ_I at any point $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$. In other words, we prove that ζ_I is an entire function over $X(\mathbb{C}_\infty) \setminus \{\infty\}$, as proven by Chung, Ngo Dac, and Pellarin in the case $I = A$ ([CNP23][Lemma 1.1]).

From now on, for any series $s \in K[[u^{\frac{1}{q^n}}]][u^{-1}]$ for some n, we denote by $s_{(i)}$ the coefficient of u^i, and by $v(s)$ the least element in $\frac{1}{q^n}\mathbb{Z}$ such that $s_{(v(s))} \neq 0$. By v we also denote the valuation on $\mathbb{C}_\infty$ with the property $v(u) = 1$. Finally, recall that d is defined as the degree of the ideal I.

Proposition 4.3.2. *For any point $P \in X(\mathbb{C}_\infty)$ different from ∞, which corresponds to a morphism of $\mathbb{F}_q$-algebras $\chi_P : A \to \mathbb{C}_\infty$, the sequence $(\zeta_{I,m}(P))_m$ and the series $\sum_{i \geq 0} \chi_P\left((\zeta_I)_{(i)}\right) u^i$ converge to the same element of $\mathbb{C}_\infty$.*

To prove the proposition, we first need some results on the coefficients $\left((\zeta_I)_{(i)}\right)_i$.

Lemma 4.3.3. *For all integers $i \geq 0$, we have* $\deg((\zeta_{I,m})_{(i)}) \leq \log_q(i+1) + g + d + 1$ *for $m \geq 0$.*

Proof. Recall the definition of j_m, and that $m + d + 1 \leq j_m \leq m + g + d$, from Remark 4.1.2. The coefficients of $\zeta_{I,0}$ have degree $j_0 \leq g + d$, so the lemma holds for $m = 0$. Since $v(\zeta_{I,m}) = j_0$ for all $m \geq 0$, the coefficient $(\zeta_{I,m})_{(0)}$, is nonzero if and only if $I = A$; in that case, it's equal to $\sum_{a \in \mathbb{F}_q^\times} a^{-1} \otimes a = -1$, and its valuation is 0, so the lemma also holds for $i = 0$. Let's prove the lemma for $i \geq 1$, $m \geq 1$.

We claim that it suffices to prove the following inequality, for all $m \geq 1$:

$$v\left(-\sum_{a \in I(j_m)} a^{-1} \otimes a\right) = v(\zeta_{I,m} - \zeta_{I,m-1}) \geq q^{m-1}.$$

If the inequality is true for $m \geq 1$, fix $i > 0$, and set $n := \lfloor \log_q(i) \rfloor + 1$, so that $q^{n-1} \leq i < q^n$; then, for $m \geq n$:

$$\deg((\zeta_{I,m})_{(i)}) = \deg\left(\left(\sum_{k=0}^{m} \zeta_{I,k} - \zeta_{I,k-1}\right)_{(i)}\right) = \deg\left(\left(\sum_{k=0}^{n} \zeta_{I,k} - \zeta_{I,k-1}\right)_{(i)}\right) \leq j_n,$$

which is at most $n + g + d = \lfloor \log_q(i) \rfloor + g + d + 1 \leq \log_q(i+1) + g + d + 1$.

The argument that follows is similar to that of [CNP23][Thm. 2.4]. Recall that by Proposition 4.1.5, for all $m \geq 0$:

$$\mathrm{Div}(\zeta_{I,m}) = \Xi^{(1)} + \cdots + \Xi^{(m)} + I + W_m - j_m \infty$$

for some effective divisor W_m with $h^0(W_m) = 1$. On one hand, for $m \geq 1$, $\zeta_{I,m} - \zeta_{I,m-1}$ has only one pole, of degree at most j_m, at ∞, and has I and $\Xi^{(1)}, \dots, \Xi^{(m-1)}$ among its zeroes; moreover, $\zeta_{I,m}(\Xi) - \zeta_{I,m-1}(\Xi) = 1 - 1 = 0$. On the other hand, since

$$1 = h^0(W_m) = h^0(j_m\infty - I - \Xi - \dots - \Xi^{(m-1)}) = h^0(j_m\infty - I - \Xi^{(1)} - \dots - \Xi^{(m)}),$$

the remaining set of zeroes coincides with the divisor $W_m^{(-1)}$, hence $\zeta_{I,m} - \zeta_{I,m-1} = (\lambda \otimes 1)\zeta_{I,m}^{(-1)}$ for some $\lambda \in K_\infty$. Since $\left(\zeta_{I,m}^{(-1)}\right)(\Xi^{(-1)}) = (\zeta_{I,m}(\Xi))^{\frac{1}{q}} = 1$, we get $\lambda = (\zeta_{I,m} - \zeta_{I,m-1})(\Xi^{(-1)})$.

If we fix $b \in I(j_m)$ with $\operatorname{sgn}(b) = 1$, we get the following:

$$\left((\zeta_{I,m} - \zeta_{I,m-1})(\Xi^{(-1)})\right)^q = -\sum_{a \in I(j_m)} a^{1-q} = -\sum_{\substack{a \in I(j_m) \\ \operatorname{sgn}(a)=1}} a^{1-q} = \sum_{c \in I(<j_m)} (b+c)^{1-q}$$

$$= b^{1-q} \sum_{c \in I(<j_m)} \sum_{i \geq 0} \binom{1-q}{i} \frac{c^i}{b^i} = b^{1-q} \sum_{i \geq 0} b^{-i} \binom{1-q}{i} \sum_{c \in I(<j_m)} c^i.$$

On the other hand, if we fix a basis $\{a_i\}_{i=1,\dots,m}$ of $I(<j_m)$, by Lemma 4.1.6, we have:

$$\sum_{c \in I(<j_m)} c^i = S_{m,i}(a_1, \dots, a_m) = 0 \quad \forall i < q^m - 1.$$

As elements of $\mathcal{O}_{K_\infty} \cong \mathbb{F}_q[[u]] \subseteq K[[u]]$, $v\left(\frac{c}{b}\right) \geq 1$ for all $c \in I(<j_m)$, and $v(b^{-1}) = j_m$, so we get:

$$q \cdot v\left((\zeta_{I,m} - \zeta_{I,m-1})(\Xi^{(-1)})\right) = (1-q)v(b) + v\left(\sum_{i \geq q^m-1} \binom{1-q}{i} \sum_{c \in I(<j_m)} \left(\frac{c}{b}\right)^i\right)$$

$$\geq (1-q)v(b) + \min_{\substack{c \in I(<j_m) \\ i \geq q^m-1}} \left\{i \cdot v\left(\frac{c}{b}\right)\right\} \geq j_m(q-1) + q^m - 1.$$

Since $\zeta_{I,m}^{(-1)} \in K[[u^{\frac{1}{q}}]]$, we deduce that, for $m \geq 1$:

$$v\left(\zeta_{I,m} - \zeta_{I,m-1}\right) = v\left(\left((\zeta_{I,m} - \zeta_{I,m-1})(\Xi^{(-1)}) \otimes 1\right)\zeta_{I,m}^{(-1)}\right) \geq j_m \frac{q-1}{q} + q^{m-1} - \frac{1}{q} \geq q^{m-1}.$$

□

Remark 4.3.4. The previous proof makes use of the fact that $I(m)$ is an $\mathbb{F}_q$-vector space of dimension 1 for $m \gg 0$, which is not true if we do not assume ∞ to be $\mathbb{F}_q$-rational.

Lemma 4.3.5. *Fix a point $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$, corresponding to a map $\chi_P : A \to \mathbb{C}_\infty$. There is a nonnegative real constant k_P such that $v(\chi_P(a)) \geq -k_P \deg(a)$ for all $a \in A$.*

Proof. Since A is a finitely generated $\mathbb{F}_q$-algebra, we can pick elements $a_1, \dots, a_n \in A$ such that their finite products generate A as an $\mathbb{F}_q$-vector space. Without loss of generality, $\deg(a_i) > 0$ for all i and we can define the following nonnegative real number:

$$k_P := \max\left\{0, \max_{1 \leq i \leq n}\left\{\frac{-v(\chi_P(a_i))}{\deg(a_i)}\right\}\right\},$$

so that $v(\chi_P(a_i)) \geq -k_P \deg(a_i)$ for all i. Given a nonzero $a \in A$, we prove by induction on $\deg(a)$ that $v(\chi_P(a)) \geq -k_P \deg(a)$.

If $\deg(a) = 0$, i.e. $a \in \mathbb{F}_q$, the claim is trivially true. If $\deg(a) > 0$ there is a product $a' := \lambda \prod_i a_i^{e_i}$, with $\lambda \in \mathbb{F}_q$, of the same degree and sign, hence $\deg(a - a') < \deg(a)$. We have:

$$v(\chi_P(a - a')) \geq -k_P \deg(a - a') \geq -k_P \deg(a) \text{ by inductive hypothesis, since } k_P \geq 0;$$
$$v(\chi_P(a')) = \sum_i e_i \cdot v(\chi_P(a_i)) \geq -k_P \sum_i e_i \cdot \deg(a_i) = -k_P \deg(a') = -k_P \deg(a).$$

Hence, $v(\chi_P(a)) \geq \min\{v(\chi_P(a')), v(\chi_P(a - a'))\} \geq -k_P \deg(a)$. □

From the previous lemmas we can deduce the following.

Lemma 4.3.6. *For all $k \geq 0$, for all $i \in \frac{1}{q^k}\mathbb{N}$, $\deg((\zeta_I^{(-k)})_{(i)}) \leq \log_q(i+1) + k + g + d + 1$. For all points $P \in X(\mathbb{C}_\infty) \setminus \{\infty\}$, corresponding to maps $\chi_P : A \to \mathbb{C}_\infty$, for all $k \geq 0$, the following series converges:*

$$\sum_{i \geq 0} \chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i.$$

Proof. The first part of the statement for $k = 0$ follows from the inequality of Lemma 4.3.3, using the fact that for all i the sequence $((\zeta_{I,m})_{(i)})_m$ is eventually equal to $(\zeta_I)_{(i)}$. For $k > 0$ and $i \in \frac{1}{q^k}\mathbb{N}$, we get:

$$\deg\left((\zeta_I^{(-k)})_{(i)}\right) = \deg\left((\zeta_I)_{(iq^k)}\right) \leq \log_q(iq^k + 1) + g + d + 1 \leq \log_q(i+1) + k + g + d + 1.$$

Let's define k_P as in Lemma 4.3.5. Then, for all $i > 0$ we have:

$$v\left(\chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i\right) \geq -k_P \deg\left((\zeta_I^{(-k)})_{(i)}\right) + i \geq i - k_P \log_q(i+1) - k_P(k + g + d + 1),$$

which tends to infinity for $i \to \infty$, proving the convergence of $\sum_{i \geq 0} \chi_P\left((\zeta_I^{(-k)})_{(i)}\right) u^i$. □

Finally we can prove Proposition 4.3.2.

Proof of Proposition 4.3.2. Define k_P as in Lemma 4.3.5. For $m \geq 0$, by Lemma 4.3.3 we have:

$$v(\zeta_I - \zeta_{I,m}) = v\left(\sum_{m' \geq m} \zeta_{I,m'+1} - \zeta_{I,m'}\right) \geq \min_{m' \geq m} v\left(\zeta_{I,m'+1} - \zeta_{I,m'}\right) \geq q^m.$$

For all $i \geq q^m$, by Lemma 4.3.6, we have:

$$\begin{aligned}\deg\left((\zeta_I - \zeta_{I,m})_{(i)}\right) &\leq \max\left\{\deg\left((\zeta_I)_{(i)}\right), \deg\left((\zeta_{I,m})_{(i)}\right)\right\} \\ &\leq \max\{\log_q(i+1) + g + d + 1, j_m\} = \log_q(i+1) + g + d + 1,\end{aligned}$$

since $j_m \leq m + g + d + 1$ and $m \leq \log_q(i+1)$. In particular:

$$\begin{aligned}v\left(\sum_i \chi_P\left((\zeta_I - \zeta_{I,m})_{(i)}\right) u^i\right) &= v\left(\sum_{i \geq q^m} \chi_P\left((\zeta_I - \zeta_{I,m})_{(i)}\right) u^i\right) \\ &\geq \min_{i \geq q^m}\left\{i - k_P \cdot \deg\left((\zeta_I - \zeta_{I,m})_{(i)}\right)\right\} \\ &\geq \min_{i \geq q^m}\left\{i - k_P(\log_q(i+1) + g + d + 1)\right\},\end{aligned}$$

which tends to infinity for $m \to \infty$. By Proposition 3.3.17, $\zeta_{I,m}(P) = \sum_i \chi_P\left((\zeta_{I,m})_{(i)}\right) u^i$, hence we get that

$$\lim_m \zeta_{I,m}(P) - \sum_{i \geq 0} \chi_P\left((\zeta_I)_{(i)}\right) u^i = \lim_m \left(\sum_i \chi_P\left((\zeta_{I,m} - \zeta)_{(i)}\right) u^i \right) = 0. \qquad \square$$

Definition 4.3.7. We define the evaluation of ζ_I at P as $\zeta_I(P) := \sum_i (\zeta_I)_{(i)}(P) u^i$.

Corollary 4.3.8. *For all $i \geq 1$, we have $\zeta_I(\Xi^{(i)}) = 0$. Similarly, for all $k \geq 0$, for all $i \geq 1$, $\sum_j \chi_{\Xi^{(i-k)}}((\zeta_I^{(-k)})_{(j)}) u^j = 0$ (where j varies in $\frac{1}{q^k}\mathbb{N}$).*

Proof. For the first identity we use that, for all $i \geq 1$, $\zeta_{I,m}(\Xi^{(i)}) = 0$ for $m \gg 0$. For the second identity, note that

$$\left(\sum_{j \in \frac{1}{q^k}\mathbb{N}} \chi_{\Xi^{(i-k)}}((\zeta_I^{(-k)})_{(j)}) u^j \right)^{q^k} = \sum_{j \in \mathbb{N}} \chi_{\Xi}^{(i)}((\zeta_I)_{(j)}) u^j = 0. \qquad \square$$

4.3.2 Adjoint Drinfeld modules and adjoint shtuka functions

From now on, in this section we use the following notation: $V_* := V_{\bar{I},*}$, $f_* := f_{\bar{I},*}$, $\zeta := (\gamma_I \otimes 1)\zeta_I$, with γ_I defined as in Corollary 4.1.14, so that $\zeta^{(-1)} = f_* \zeta$.

Remark 4.3.9. There is an anti-isomorphism between the noncommutative $\mathbb{F}_q$-algebras $\mathbb{C}_\infty[[\tau]]$ and $\mathbb{C}_\infty[[\tau^{-1}]]$, sending $h = \sum_i h_i \tau^i \in \mathbb{C}_\infty[[\tau]]$ to $h^* := \sum_i \tau^{-i} h_i$.

Definition 4.3.10. Let $\phi : A \to \mathbb{C}_\infty[\tau]$ be a Drinfeld module. The adjoint Drinfeld module $\phi^* : A \to \mathbb{C}_\infty[\tau^{-1}]$ is the $\mathbb{F}_q$-algebra homomorphism sending $a \in A$ to ϕ_a^*.

Proposition 4.3.11 shows a connection between adjoint Drinfeld modules, adjoint shtuka functions, and zeta functions, which is meant to mirror the correspondence between Drinfeld modules, shtuka functions, and special functions (cf. [Tha93][Eq.(∗∗)], [And94][Eq.(46)]).

Afterwards, we present some basic definitions and results concerning the coefficients of exponential and logarithmic functions (see for example [Gos98]) to prove the interesting Proposition 4.3.22. On the surface the proposition resembles a log-algebraicity result, and could be linked to this rich branch of research (see for example [And94], [And96], [ANT17b]); on the other hand, it encourages a greater focus on the adjoint exponential function, whose kernel was already studied in works such as [Poon96], and is partly carried out in the following chapter.

Proposition 4.3.11. *Set $s_m := \prod_{i=0}^{m-1} f_*^{(-i)}$ for all nonnegative integers m. The set $\{s_m\}_{m \geq 0}$ is a basis of the $\mathbb{C}_\infty \otimes \mathbb{F}_q$-vector space $H^0(X_{\mathbb{C}_\infty} \setminus \{\infty\}, V_*^{(1)})$.*

For all $a \in A$, $1 \otimes a$ can be expressed as $\sum_{i=0}^{\deg(a)} (a_i \otimes 1) s_i$ with $a_i^{q^i} \in K_\infty$, and the function $\phi^* : A \to \mathbb{C}_\infty[\tau^{-1}]$ sending a to $\sum_i a_i \tau^{-i}$ is the adjoint of a normalized Drinfeld module ϕ of rank 1.

Finally, for all $a \in A$, $(\phi_a^* \otimes 1)(\zeta) = (1 \otimes a)\zeta$.

Proof. Since $H^0(X_{\mathbb{C}_\infty} \setminus \{\infty\}, V_*^{(1)}) = \bigcup_{m \geq 0} H^0(X_{\mathbb{C}_\infty}, V_*^{(1)} + m\infty)$, for the first part we just need to prove that, for all $m \geq 0$, $s_m \in H^0(X_{\mathbb{C}_\infty} \setminus \{\infty\}, V_*^{(1)})$ and it has a pole of multiplicity exactly m at ∞; using that $\mathrm{Div}(f_*^{(-i)}) = V_*^{(-i)} - V_*^{(1-i)} + \Xi^{(-i)} - \infty$, we get:

$$\mathrm{Div}(s_m) = \mathrm{Div}\left(\prod_{i=0}^{m-1} f_*^{(-i)} \right) = V_*^{(1-m)} - V_*^{(1)} + \sum_{i=0}^{m-1} \Xi^{(-i)} - m\infty.$$

If we fix $a \in A$ of degree m, $1 \otimes a \in H^0(X_{\mathbb{C}_\infty}, V_*^{(1)} + m\infty)$, hence it can be expressed as a sum $\sum_{i=0}^m (a_i \otimes 1)s_i$. Moreover, if we twist k times and evaluate at Ξ for all $0 \le k \le m$ we get the following triangular system of equations in the variables $(a_i)_i$:

$$\left\{ a = \sum_{i=0}^{k} \left(a_i^{q^k} \prod_{j=k-i}^{k} f_*^{(j)}(\Xi) \right) \right\}_k$$
$$\implies \left\{ a_k^{q^k} = \left(\prod_{j=0}^{k} f_*^{(j)}(\Xi) \right)^{-1} \left(a - \sum_{i=0}^{k-1} a_i^{q^k} \prod_{j=k-i}^{k} f_*^{(j)}(\Xi) \right) \right\}_k .$$

From this system we can deduce that $a_0 = a$ and, since $f_*^{(j)}(\Xi) \in K_\infty$ for all $j \ge 0$, that $a_k^{q^k} \in K_\infty$ for all k. Finally, since $\deg(a) = \deg(s_m)$, and $\operatorname{sgn}(f_*^{(i)}) = \operatorname{sgn}(f_*) = 1$ for all $i \ge 0$, the sign of $s_m^{(i)}$ is also 1 for all m, i, and we have:

$$\operatorname{sgn}(a) = \operatorname{sgn}\left(\sum_{i=0}^{m} (a_i^{q^m} \otimes 1) s_i^{(m)} \right) = \operatorname{sgn}((a_m^{q^m} \otimes 1) s_m^{(m)}) = a_m^{q^m} \operatorname{sgn}(s_m^{(m)}) = a_m^{q^m},$$

so $a_m = \operatorname{sgn}(a)$. For all $a \in A$, write $\phi_a^* := \sum_i a_i \tau^{-i}$. Since for all $k \ge 0$ and for all $a \in A$ we have $\zeta s_k = \zeta^{(-k)}$ and $1 \otimes a = (1 \otimes a)^{(-k)} = \sum_i (a_i^{\frac{1}{q^k}} \otimes 1) s_i^{(-k)}$, we get the following equations for all $k \ge 0$ and $a, b \in A$:

$$\begin{aligned}
(1 \otimes a)\zeta^{(-k)} &= \sum_i (a_i^{\frac{1}{q^k}} \otimes 1)(s_i \zeta)^{(-k)} = \sum_i (a_i^{\frac{1}{q^k}} \otimes 1)\zeta^{(-k-i)} = \tau^{-k} \circ (\phi_a^* \otimes 1)(\zeta); \\
(\phi_{ab}^* \otimes 1)(\zeta) &= (1 \otimes a)\left((1 \otimes b)\zeta\right) = (1 \otimes a)\left((\phi_b^* \otimes 1)(\zeta)\right) \\
&= (\phi_b^* \otimes 1)\left((1 \otimes a)\zeta\right) = (\phi_b^* \otimes 1)\left((\phi_a^* \otimes 1)(\zeta)\right) = (\phi_b^* \phi_a^* \otimes 1)(\zeta).
\end{aligned}$$

Since the elements $(\zeta^{(-i)})_{i \ge 0} = (\zeta s_i)_{i \ge 0}$ are all $\mathbb{C}_\infty \otimes \mathbb{F}_q$-linearly independent, we have the equality $\phi_{ab}^* = \phi_b^* \circ \phi_a^*$. Together with the fact that $\deg(\phi_a^*) = \deg(a)$ and $a_{\deg(a)} = \operatorname{sgn}(a)$, this means that the function $\phi := (\phi^*)^* : A \to K_\infty[\tau]$ is a normalized Drinfeld module of rank 1. $\square$

Definition 4.3.12. Let ϕ^*, f_* be as in the previous proposition. Then f_* is said to be the adjoint shtuka function *associated* to ϕ.

From this point onwards, ϕ and ϕ^* are defined as in Proposition 4.3.11.

Definition 4.3.13. We choose a nonzero element of least norm $\tilde{\pi}_\phi \in \Lambda_\phi$, and we call it the *fundamental period* of Λ_ϕ. We denote by $\Lambda := \tilde{\pi}_\phi^{-1} \Lambda_\phi$.

Remark 4.3.14. Our choice of $\tilde{\pi}_\phi$ is determined up to a factor in $\mathbb{F}_q^\times$.

Since $\operatorname{rk}(\Lambda_\phi) = 1$, all elements of Λ_ϕ are of the form $c\tilde{\pi}_\phi$ for some $c \in K$; in particular, $\Lambda \subseteq K$ is a fractional ideal.

More precisely, $\Lambda \subseteq K$ is the unique fractional ideal isomorphic to Λ_ϕ such that its nonzero elements of least norm are the constant functions $\mathbb{F}_q^\times$.

Let's denote by $\exp_\phi = \sum_i e_i \tau^i \in \mathbb{C}_\infty[[\tau]]$ the exponential function associated to ϕ. We define its adjoint as $\exp_\phi^* := \sum_i \tau^{-i} e_i \in \mathbb{C}_\infty[[\tau^{-1}]]$.

Remark 4.3.15. Since $\exp_\phi \circ (a\tau^0) = \phi_a \circ \exp_\phi$ for all $a \in A$, we easily deduce the following identities in $\mathbb{C}_\infty[[\tau^{-1}]]$ for all $a \in A$:

$$a \exp_\phi^* = \exp_\phi^* \circ \phi_a^*.$$

Remark 4.3.16. If we write $\phi_a = \sum_j a_j \tau^j \in H[\tau]$ for some $a \in A \setminus \mathbb{F}_q$ (with $a_0 = a$), the equation $\exp_\phi \circ (a\tau^0) = \phi_a \circ \exp_\phi$ becomes:

$$\sum_k (e_k a^{q^k})\tau^k = \sum_k \left(\sum_{i+j=k} a_j e_i^{q^j} \right) \tau^k \Rightarrow e_k(a^{q^k} - a) = \sum_{i=0}^{k-1} a_{k-i} e_i^{q^{k-i}}.$$

Since $e_0 = 1$, we get that $e_k \in H$ for all $k \geq 0$ by induction.

Definition 4.3.17. For any projective A-submodule $L \subseteq \mathbb{C}_\infty$ of finite rank, we define, for all $k \geq 1$:

$$S_k(L) := \sum_{\substack{\lambda_1, \ldots, \lambda_k \in L \setminus \{0\} \\ i \neq j \Rightarrow \lambda_i \neq \lambda_j}} (\lambda_1 \cdots \lambda_k)^{-1}; \qquad P_k(L) := \sum_{\lambda \in L \setminus \{0\}} \lambda^{-k}.$$

We also set $S_0(L) := 1$ and $P_0(L) := -1$.

Remark 4.3.18. The infinite product $\exp_L(x) := x \prod_{\lambda \in L \setminus \{0\}} \left(1 - \frac{x}{\lambda}\right)$ converges in $\mathbb{C}_\infty[[x]]$ (see e.g. [Gos98][Section 4.2]), by absolute convergence we can expand the product and rearrange the terms of the series, so we get $\exp_L(x) = \sum_{i \geq 0} S_i(L) x^{i+1}$; in particular, if $i+1$ is not a power of q, $S_i(L) = 0$.

Remark 4.3.19. Note that in the summation that defines $S_{q^i-1}(\tilde{\pi}_\phi \Lambda)$ there is a unique summand of greatest norm, given by the inverse of the product of the $q^i - 1$ nonzero elements of lower norm of $\tilde{\pi}_\phi \Lambda$. Since the elements of $\mathbb{F}_q^\times$ are the nonzero elements of lowest degree of Λ, this product has valuation at least:

$$\sum_{j=0}^{i-1} (q^{j+1} - q^j)(j + v(\tilde{\pi}_\phi)) = iq^i - \left(\sum_{j=0}^{i-1} q^j \right) + q^i v(\tilde{\pi}_\phi) \geq (i - 1 + v(\tilde{\pi}_\phi))q^i.$$

In particular, since $\lim_i v\left(e_i^{q^{-i}}\right) = \lim_i \frac{1}{q^i} v\left(S_{q^i-1}(\tilde{\pi}_\phi \Lambda)\right) = \infty$, $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an entire function with an infinite radius of convergence, which is well known, and $\exp_\phi^* : \mathbb{C}_\infty \to \mathbb{C}_\infty$, while not being a power series, is continuous, converges everywhere, and sends 0 to 0 (see also [Poon96][Prop. 1]); moreover, $\lim_{z \to 0} \exp_\phi^*(z) z^{-1} = 1$.

Remark 4.3.20. Since $\exp_\phi$ and $\exp_\phi^*$ are continuous $\mathbb{F}_q$-linear endomorphisms of $\mathbb{C}_\infty$, they can be extended uniquely to continuous $\mathbb{F}_q \otimes K$-linear endomorphisms of $\mathbb{C}_\infty \hat{\otimes} K$.

The following is a well known lemma (see [Gek88][Eqq. 2.8,2.9]).

Lemma 4.3.21. *For any projective A-submodule $L \subseteq \mathbb{C}_\infty$, the inverse of* $\exp_L$*, denoted by* $\log_L$*, is* $-\sum_i P_{q^i-1}(L)\tau^i$.

Proof. For all $i \geq 1$, $-iS_i(L) = \sum_{j=0}^{i-1} (-1)^{i-j} S_j(L) P_{i-j}(L)$ by Newton's identities. Setting $i = q^k - 1$ with $k \geq 1$, since $S_j(L) = 0$ if $j+1$ is not a power of q, we get:

$$S_{q^k-1}(L) = \sum_{j=0}^{k-1} S_{q^j-1}(L) P_{q^k-q^j}(L) = \sum_{j=0}^{k-1} S_{q^j-1}(L) (P_{q^{k-j}-1}(L))^{q^j}.$$

In particular:

$$\begin{aligned}\exp_L \circ \left(-\sum_{i\geq 0} P_{q^i-1}(L)\tau^i \right) &= \sum_{k\geq 0} \left(-\sum_{j=0}^{k} S_{q^j-1}(L)(P_{q^{k-j}-1}(L))^{q^j} \right) \tau^k \\ &= -S_0(L)P_0(L) = 1.\end{aligned}$$

The uniqueness of right inverses proves the thesis. □

Proposition 4.3.22. *The following functional identity holds in $\mathbb{C}_\infty \hat{\otimes} K$:*

$$\exp_\phi^*(\zeta) = 0.$$

Proof. By Remark 4.3.15, for all $a \in A$ we have $\exp_\phi^* \circ \phi_a^* = (a \otimes 1) \exp_\phi^*$ as endomorphisms of $\mathbb{C}_\infty \hat{\otimes} K$; by Proposition 4.3.11, $\phi_a^*(\zeta) = (1 \otimes a)\zeta$. Hence, for all $a \in A$:

$$0 = \exp_\phi^*(0) = \exp_\phi^*(\phi_a^*(\zeta) - (1 \otimes a)\zeta) = (a \otimes 1 - 1 \otimes a) \exp_\phi^*(\zeta).$$

For $a \notin \mathbb{F}_q$, $a \otimes 1 - 1 \otimes a$ is invertible in $\mathbb{C}_\infty \hat{\otimes} K$, with inverse $\sum_{i\geq 0} a^{-i-1} \otimes a^i$, so we get the thesis. □

4.3.3 The fundamental period $\tilde{\pi}_\phi$

Finally, in this subsection we are able to link the zeta function ζ_I and the fundamental period $\tilde{\pi}_\phi$. Fix an element $a_I \in I \setminus \{0\}$ of least degree.

Proposition 4.3.23. *The A-modules $a_I^{-1} I$ and Λ coincide as submodules of $\mathbb{C}_\infty$.*

Proof. Since the nonzero elements of least degree of both $a_I^{-1} I$ and Λ are $\mathbb{F}_q^\times$, it suffices to show that I and Λ are isomorphic. Let's first give an intuitive rundown of the proof.

For all $n \geq 0$, for all $k \geq 0$, if $n < k$ then $\zeta^{(-k)}(\Xi^{(-n)}) = (\zeta(\Xi^{(k-n)}))^{\frac{1}{q^k}} = 0$ by Corollary 4.3.8, while if $n \geq k$ then:

$$\begin{aligned}\zeta^{(-k)}(\Xi^{(-n)}) &= \gamma_I^{\frac{1}{q^k}} \zeta_I^{(-k)}(\Xi^{(-n)}) = -\gamma_I^{\frac{1}{q^k}} \sum_{a\in I\setminus\{0\}} a^{\frac{1}{q^n} - \frac{1}{q^k}} \\ &= -\left(\gamma_I \sum_{a\in I\setminus\{0\}} \left(\frac{a}{\gamma_I}\right)^{1-q^{n-k}} \right)^{\frac{1}{q^n}} = -\left(\gamma_I P_{q^{n-k}-1}(\gamma_I^{-1} I)\right)^{\frac{1}{q^n}} .\end{aligned}$$

Since $\exp_\phi^*(\zeta) = 0$ by Proposition 4.3.22, evaluating $\exp_\phi^*(\zeta)$ at $\Xi^{(-n)}$ we should get:

$$\begin{aligned}0 = \exp_\phi^*(\zeta)(\Xi^{(-n)}) &= \sum_{k\geq 0} S_{q^k-1}(\Lambda_\phi)^{\frac{1}{q^k}} \zeta^{(-k)}(\Xi^{(-n)}) \\ &= \sum_{k\geq 0} S_{q^k-1}(\Lambda_\phi)^{\frac{1}{q^k}} \gamma_I^{\frac{1}{q^k}} \zeta_I^{(-k)}(\Xi^{(-n)}) \\ &= -\sum_{k=0}^{n} S_{q^k-1}(\Lambda_\phi)^{\frac{1}{q^k}} \left(\gamma_I P_{q^{n-k}-1}(\gamma_I^{-1} I)\right)^{\frac{1}{q^n}} \\ &= -\left(\gamma_I \sum_{0\leq k\leq n} P_{q^{n-k}-1}(\gamma_I^{-1} I) S_{q^k-1}(\Lambda_\phi)^{q^{n-k}} \right)^{\frac{1}{q^n}} ,\end{aligned}$$

which by Lemma 4.3.21 implies that $\log_{\gamma_I^{-1}I} \circ \exp_\phi = 1$. In particular, $\exp_\phi = \exp_{\gamma_I^{-1}I}$, therefore their zero loci are the same, which means that $\gamma_I^{-1}I = \Lambda_\phi = \tilde{\pi}_\phi\Lambda$.

The previous reasoning is not rigorous only when it assumes that evaluation at $\Xi^{(-n)}$ commutes with the expansion of $\exp_\phi^*(\zeta)$, therefore to prove the theorem it suffices to show the following identity:

$$\sum_{k=0}^{n} (S_{q^k-1}(\Lambda_\phi)\gamma_I)^{\frac{1}{q^k}} \zeta_I^{(-k)}(\Xi^{(-n)}) = 0.$$

For all $k \in \mathbb{N}$, set $c_k := (S_{q^k-1}(\tilde{\pi}_\phi\Lambda)\gamma_I)^{\frac{1}{q^k}}$; by Remark 4.3.16 and Corollary 4.1.14, $c_k \in \overline{\mathbb{F}_q}((u^{\frac{1}{q^k}}))$. For all $k \leq m$ we can write the following:

$$\begin{aligned} c_k &= \sum_{i \in \frac{1}{q^m}\mathbb{Z}} \lambda_{k,i} u^i \in \overline{\mathbb{F}_q}\left(\left(u^{\frac{1}{q^m}}\right)\right) \text{ with } \lambda_{k,i} \in \overline{\mathbb{F}_q} \\ \zeta_I^{(-k)} &= \sum_{i \in \frac{1}{q^k}\mathbb{Z}} \left(\zeta_I^{(-k)}\right)_{(i)} u^i \in K\left(\left(u^{\frac{1}{q^m}}\right)\right) \text{ with } \left(\zeta^{(-k)}\right)_{(i)} \in K. \end{aligned}$$

By Lemma 4.3.6 and Remark 4.3.19 respectively, we have the following inequalities for all $i \in \frac{1}{q^k}\mathbb{N}$, for all $k \in \mathbb{N}$:

$$\begin{aligned} \deg\left((\zeta_I^{(-k)})_{(i)}\right) &\leq \log_q(i+1) + k + g + d + 1, \\ v(c_k) &\geq k - 1 + v(\tilde{\pi}_\phi) + \min(\{v(\gamma_I), 0\}) =: k'. \end{aligned}$$

Fix a positive integer n. Since $\exp_\phi^*(\zeta) = 0$, for any arbitrarily large N we can choose a positive integer $m \geq n$ such that $v\left(\sum_{k=0}^m c_k\zeta_I^{(-k)}\right) \geq N$.

Let's rearrange $\sum_{k=0}^m c_k\zeta_I^{(-k)}$, with the indexes i and j varying in $\frac{1}{q^m}\mathbb{Z}$:

$$\begin{aligned} \sum_{k=0}^{m}\sum_{j \geq k'} \lambda_{k,j}\zeta_I^{(-k)}u^j &= \sum_{k=0}^{m}\sum_{j \geq k'} \lambda_{k,j} \sum_{i \geq 0} \left(\zeta_I^{(-k)}\right)_{(i)} u^{i+j} \\ &= \sum_{i \geq 0}\left(\sum_{k=0}^{m}\sum_{j=k'}^{i} \lambda_{k,j}\left(\zeta_I^{(-k)}\right)_{(i-j)}\right)u^i. \end{aligned}$$

Since $v\left(\sum_{k=0}^m c_k\zeta_I^{(-k)}\right) \geq N$, we get that, for $i \in \frac{1}{q^m}\mathbb{Z}$ and $i < N$:

$$\sum_{k=0}^{m}\sum_{j=k'}^{i} \lambda_{k,j}\left(\zeta_I^{(-k)}\right)_{(i-j)} = 0.$$

Using this result and Lemma 4.3.6, the evaluation $\sum_{k=0}^m c_k\zeta_I^{(-k)}(\Xi^{(-n)})$ can be rearranged as follows:

$$\begin{aligned} \sum_{k=0}^{m}\sum_{j \geq k'} \lambda_{k,j}\zeta_I^{(-k)}(\Xi^{(-n)})u^j &= \sum_{k=0}^{m}\sum_{j \geq k'} \lambda_{k,j} \sum_{i \geq 0} \left(\zeta_I^{(-k)}\right)_{(i)} (\Xi^{(-n)})u^{i+j} \\ &= \sum_{i \geq 0}\left(\sum_{k=0}^{m}\sum_{j=k'}^{i} \lambda_{k,j}\left(\zeta_I^{(-k)}\right)_{(i-j)}\right)(\Xi^{(-n)})u^i \\ &= \sum_{i \geq N}\left(\sum_{k=0}^{m}\sum_{j=k'}^{i} \lambda_{k,j}\left(\zeta_I^{(-k)}\right)_{(i-j)}\right)(\Xi^{(-n)})u^i. \end{aligned}$$

For $i-j,k\geq 0$, since $j\geq k'\geq v(\tilde{\pi})+\min(\{v(\gamma_I),0\})-1$, and since $\log_q(x)\leq x$ for all $x>0$, $\deg\left((\zeta_I^{(-k)})_{(i-j)}\right)$ is bounded from above by:

$$\log_q(i-j+1)+k+g+d+1\leq i+k+g+d+3-v(\tilde{\pi})-\min(\{v(\gamma_I),0\})=: i+C,$$

so each summand has valuation at least $i-\frac{i+C}{q^n}\geq N-\frac{N+C}{q^n}$, which tends to infinity as N tends to infinity. Since $m=m(N)$ depends on N and tends to infinity as N does, we have:

$$0=\lim_{N\to\infty}\sum_{k=0}^{m(N)}c_k\zeta_I^{(-k)}(\Xi^{(-n)})=\lim_{m\to\infty}\sum_{k=0}^{m}c_k\zeta_I^{(-k)}(\Xi^{(-n)})=\sum_{k=0}^{n}c_k\zeta_I^{(-k)}(\Xi^{(-n)}),$$

where we used that $\zeta_I^{(-k)}(\Xi^{(-n)})=0$ for $k>n$ by Corollary 4.3.8. This concludes the proof. ☐

Proposition 4.3.24. *The following identity holds in $\mathbb{C}_\infty\hat{\otimes}K$:*

$$\frac{\left((a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I\right)^{(-1)}}{(a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I}=f_*,$$

Proof. From the definition of γ_I we have $\frac{\zeta_I}{\zeta_I^{(1)}}=(\gamma_I\otimes 1)^{q-1}f_*^{(1)}$. Since $\Lambda=a_I^{-1}I$ and $\tilde{\pi}\Lambda=\gamma_I^{-1}I$, we deduce $\gamma_I=\frac{a_I}{\tilde{\pi}}$ up to a factor in $\mathbb{F}_q^\times$. ☐

We deduce the following well-known result (see e.g. [Gos98][Section 7.10]).

Corollary 4.3.25. *The element $\tilde{\pi}_\phi^{q-1}$ is contained in K_∞.*

It's possible to use the expansion of the series $\frac{z}{\exp_\phi(z)}\in\mathbb{C}_\infty[[z]]$ to prove that the Goss zeta value $\sum_{a\in A}a^{-s}\in\mathbb{C}_\infty$ belongs to $\tilde{\pi}_\phi^s\cdot H$ for all positive s multiples of $q-1$ (see [Gos98][Thm. 8.18.3]). As a corollary of Proposition 4.3.24, we can recover this well-known theorem in an alternative way for the partial zeta ζ_I when $s=q^k-1$ for some positive integer k.

Corollary 4.3.26. *For all positive integers k there is some $\alpha\in H^\times$ such that:*

$$\sum_{a\in I\setminus\{0\}}a^{1-q^k}=\tilde{\pi}_\phi^{q^k-1}\alpha$$

Proof. First, note that the left hand side is equal to $\zeta_I^{(k)}(\Xi)$ (and in particular is nonzero). Let's prove by induction on k that $\zeta_I^{(k)}(\Xi)\in\tilde{\pi}_\phi^{q^k}\cdot H$ for all $k\geq 0$.

For $k=0$ the result is trivial, since $\zeta_I(\Xi)=-1$.

Let's now suppose the result holds for some $k\geq 0$. Let's twist by $k+1$ the identity in Proposition 4.3.24:

$$\frac{\left((a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I\right)^{(k)}}{\left((a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I\right)^{(k+1)}}=f_*^{(k+1)};$$

since numerator and denominator on the left hand side, if evaluated at Ξ, are nonzero, the same holds for $f_*^{(k+1)}$, and since the latter is a rational function on X_H, we obtain:

$$\frac{(a_I\tilde{\pi}_\phi^{-1})^{q^k}\sum_{a\in I\setminus\{0\}}a^{1-q^k}}{(a_I\tilde{\pi}_\phi^{-1})^{q^{k+1}}\sum_{a\in I\setminus\{0\}}a^{1-q^{k+1}}}=\frac{\left((a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I\right)^{(k)}}{\left((a_I\tilde{\pi}_\phi^{-1}\otimes 1)\zeta_I\right)^{(k+1)}}(\Xi)=f_*^{(k+1)}(\Xi)\in H^\times,$$

hence by inductive hypothesis we have:

$$(a_I\tilde{\pi}_\phi^{-1})^{q^{k+1}}\zeta_I^{(k+1)}(\Xi) \in H^\times. \qquad \square$$

Equivalently to Proposition 4.3.24, we can say that $(a_I\tilde{\pi}_\phi^{-1} \otimes 1)\zeta_I \in \mathbb{C}_\infty\hat{\otimes}A$ is an eigenvector for the dual Drinfeld module ϕ^*, in the following sense.

Theorem 4.3.27. *The following identity holds in $\mathbb{C}_\infty\hat{\otimes}A$ for all $a \in A$:*

$$\phi_a^*\left((a_I\tilde{\pi}_\phi^{-1} \otimes 1)\zeta_I\right) = (1 \otimes a)(a_I\tilde{\pi}_\phi^{-1} \otimes 1)\zeta_I.$$

Proof. By Proposition 4.3.11, for all $a \in A$, $(1 \otimes a)\zeta = \phi_a^*(\zeta)$, where $\zeta = (\gamma_I \otimes 1)\zeta_I = (a_I\tilde{\pi}^{-1} \otimes 1)\zeta_I$ up to a factor in $\mathbb{F}_q^\times$. $\square$

The interpretation of $(a_I\tilde{\pi}_\phi^{-1} \otimes 1)\zeta_I$ as an eigenvector for ϕ^* is the starting point of the next chapters.

We can finally state and prove a strengthening of Theorem 4.1.9 and Theorem 4.2.6.

Theorem 4.3.28. *The following product expansion converges in $K_\infty\hat{\otimes}K$ and is equal to ζ_I:*

$$(a_I^{-1} \otimes a_I)\prod_{i\geq 0}\left((\tilde{\pi}_\phi^{1-q} \otimes 1)f_*^{(1)}\right)^{(i)}.$$

Proof. Using the Theorem 4.1.9, we deduce the following identity in $\mathcal{O}_{K_\infty}\hat{\otimes}K$:

$$\zeta_I = (a_I^{-1} \otimes a_I)\prod_{i\geq 0}\left((\lambda^{1-q} \otimes 1){f'_{\bar{I},*}}^{(1)}\right)^{(i)},$$

where $f'_{\bar{I},*}$ is a scalar multiple of $f_{\bar{I},*}$, and $\lambda \in \mathcal{O}_{K_\infty}$ is some constant. We deduce:

$$\frac{\zeta_I}{\zeta_I^{(1)}} = (a_I^{q-1} \otimes 1)(\lambda^{1-q} \otimes 1){f'_{\bar{I},*}}^{(1)}.$$

On the other hand, by Proposition 4.3.24, we know that

$$\frac{\zeta_I}{\zeta_I^{(1)}} = \left(\frac{a_I}{\tilde{\pi}_\phi} \otimes 1\right)^{q-1} f_*^{(1)},$$

hence $(\lambda^{1-q} \otimes 1){f'_{I,*}}^{(1)} = (\tilde{\pi}_\phi^{1-q} \otimes 1)f_*^{(1)}$ and we get the desired identity. $\square$

Recall Definition 4.3.13 and Remark 4.3.14: $\Lambda \subseteq K$ is a fractional ideal and, if $(d_i)_{i\geq 0}$ is the increasing sequence of degrees assumed by the elements of $\Lambda \setminus \{0\}$, we have $d_0 = 0$. Since $\Lambda \subseteq K$ is a fractional ideal and $\infty \in X(\mathbb{F}_q)$, for $m \gg 0$ $d_{i+m} = d_m + i$ for all $i \geq 0$ by Riemann-Roch; we take n to be the least integer with this property.

Theorem 4.3.29. *Let ϕ be a normalized Drinfeld module of rank 1, and denote by f_ϕ the corresponding shtuka function. Fix a uniformizer $u \in K_\infty$ and set:*

$$\alpha := u^{(1-d_n)\prod_{i=1}^{n-1} q^{(d_n-1-d_i)(q^{i-1}-q^i)}}.$$

The following is a well defined element of $(\mathbb{F}_q \otimes K)\,\mathrm{Sf}_\phi(A) \subseteq \mathbb{C}_\infty\hat{\otimes}K$:

$$\omega := (\tilde{\pi}_\phi\alpha \otimes 1)\prod_{i\geq 0}\left(\frac{(\tilde{\pi}_\phi\alpha)^{q-1} \otimes 1}{f_\phi}\right)^{(i)}.$$

Proof. Following the proof of Theorem 4.2.6, we just need to show that $(\tilde{\pi}_\phi\alpha)^{q-1}$ and f_ϕ have the same degree as elements of $K((u)) \cong K_\infty \hat{\otimes} K$. By Remark 2.2.13, the universal Anderson eigenvector ω_ϕ can be written as $\sum_i \exp_\phi(a_i^*) \otimes a_i$, where $\{a_i\}_{i\in\mathbb{N}}$ is a basis of $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ and $\{a_i^*\}_{i\in\mathbb{N}}$ is the dual basis of ${}^{K_\infty\Lambda_\phi}\!/\!_{\Lambda_\phi}$. Since Λ_ϕ has rank 1, we can fix an isomorphism of $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ with an ideal of $I \subseteq A$, and we can assume without loss of generality that the image of the sequence $(a_i)_i$ is strictly increasing in degree: by Proposition 2.3.5, the sequence $(a_i^*)_i$ is strictly decreasing in norm. Following the proof of Theorem 2.3.7, the sequence $(\|\exp_\phi(a_i^*)\|)_i$ is also strictly decreasing, and, if we call $c \in K_\infty\Lambda_\phi = \tilde{\pi}_\phi K_\infty$ the element of least norm lifting a_0^*, we know that its norm is $\|\tilde{\pi}_\phi\| q^{d_n-1}$, and that:

$$\|\exp_\phi(a_0^*)\| = \|c_0\| \prod_{\substack{\lambda\in\Lambda\setminus\{0\}\\ \deg(\lambda)<d_n}} \left\|\frac{c}{\lambda}\right\| = \|\tilde{\pi}_\phi\| q^{d_n-1} \prod_{i=1}^{n-1} q^{(d_n-1-d_i)(q^i-q^{i-1})} = \|\tilde{\pi}_\phi\|\|\alpha\|.$$

Since $f_\phi\omega_\phi = \omega_\phi^{(1)}$, we deduce that f_ϕ has the same degree in u as $(\tilde{\pi}_\phi\alpha)^{q-1}$, as elements of $K((u))$. □

4.3.4 A strengthening of the main theorem

Using the results of the previous subsection, and Remark 4.2.3, we can prove Theorem 4.3.32. It is a strengthening of Theorem 4.2.1 with an explicit proportionality constant and with a more natural dependence on the period lattice.

First, we prove the following results.

Lemma 4.3.30. *Fix a nonzero ideal $J < A$, with degree d_J. Then, for each ideal class $\bar{I} \in Cl(A)$, there is a rational function $h_{J,\bar{I}}$ on X_H with sign 1 and divisor:*

$$\mathrm{Div}(h_{J,\bar{I}}) = V_{\bar{I},*}^{(1)} + V_{\bar{J}+\bar{I}} - J - \Xi - (2g - d_J - 1)\infty.$$

Moreover, the functions $\{h_{J,\bar{I}}\}_{\bar{I}\in Cl(A)}$ are all conjugated by the action of $\mathrm{Gal}(H/K)$*.*

Proof. Fix some nonzero ideal $I < A$ with ideal class $\bar{I}$, and call d_I its degree. Define $D := J + \Xi + (2g - d_J - 1)\infty$ and consider the divisor $D - V_{\bar{I},*}^{(1)}$: we want to prove that it is linearly equivalent to $V_{\bar{J}+\bar{I}}$. First of all, its degree is g, hence it is linearly equivalent to some effective divisor W. Moreover, we have the following linear equivalences:

$$\begin{aligned}
\mathrm{red}_{K_\infty}(W) &\sim \mathrm{red}_{K_\infty}(D) - \mathrm{red}_{K_\infty}(V_{\bar{I},*}) \sim (J+I) + (g - d_J - d_I)\infty \sim \mathrm{red}_{K_\infty}(V_{\bar{J}+\bar{I}});\\
W - W^{(1)} &\sim (D - D^{(1)}) - (V_{\bar{I},*} - V_{\bar{I},*}^{(1)})^{(1)} \sim (\Xi - \Xi^{(1)}) - (\infty - \Xi)^{(1)} \sim \Xi - \infty\\
&\sim V_{\bar{J}+\bar{I}} - V_{\bar{J}+\bar{I}}^{(1)}.
\end{aligned}$$

By Proposition 3.3.25, the two conditions imply that $W = V_{\bar{J}+\bar{I}}$.

By Remark 3.3.26, the positive and negative components of the divisors $\{V_{\bar{I},*}^{(1)} + V_{\bar{I}+\bar{J}} - D\}_{\bar{I}\in Cl(A)}$ are H-rational. Moreover, by the same reasoning as Remarks 3.3.26 and 4.1.13, they are all conjugated by the action of $\mathrm{Gal}(H/K)$, therefore for all $\bar{I} \in Cl(A)$ there is a unique rational function $h_{J,\bar{I}}$ on X_H with divisor $V_{\bar{I},*}^{(1)} + V_{\bar{I}+\bar{J}} - D$ and sign 1, and they are all conjugated by $\mathrm{Gal}(H/K)$ up to a scalar factor. We just need to prove that for all $\sigma \in \mathrm{Gal}(H/K)$, $\mathrm{sgn}(h_{J,\bar{A}}^\sigma) = 1$. Fix some $a \in J$ of positive degree and sign 1, and define $s := (1 \otimes a)(1 - a^{-1} \otimes a)h_{J,\bar{A}}$: it suffices to prove that $\mathrm{sgn}(s^\sigma) = 1$ for all $\sigma \in \mathrm{Gal}(H/K)$. Since s has poles only at ∞, it can be written as a finite sum $\sum_{i=0}^{d} h_i \otimes a_i \in A_H$ for some nonnegative integer d, where the a_i's have sign 1 and strictly increasing degree, and $h_d \neq 0$. Since $\mathrm{sgn}(s) = 1$, we deduce $h_d = 1$, hence for all $\sigma \in \mathrm{Gal}(H/K)$ $\mathrm{sgn}(s^\sigma) = h_d^\sigma = 1$. □

Proposition 4.3.31. *For any pair of ideals $I, J < A$, with degrees respectively d_I and d_J, the quotient $\frac{\zeta_I}{\zeta_J}$ is a rational function on X_H, with divisor $V_{\bar{I},*}^{(1)} - V_{\bar{J},*}^{(1)} + I - J + (d_J - d_I)\infty$. Moreover,* $\mathrm{red}_u\left(\mathrm{sgn}\left(\frac{\zeta_I}{\zeta_J}\right)\right) = 1$.

Proof. Recall that for all integers $m \gg 0$ there is a rational function $\delta_{\bar{J},m}$ on X_{K_∞} with divisor $V_{\bar{J},m} + V_{\bar{J},*,m} - 2g\infty$. Since the sequence $(V_{\bar{J},m} + V_{\bar{J},*,m})_m$ converges to $V_{\bar{J}}^{(1)} + V_{\bar{J},*}^{(1)}$ in $X^{[2g]}(K_\infty)$, by Proposition 3.2.9 we can choose each $\delta_{\bar{J},m}$ so that the sequence $(\delta_{\bar{J},m})_m$ converges to $\delta_{\bar{J}}^{(1)}$ in $K((u))$. In $K((u))$, $\frac{\zeta_I}{\zeta_J}\delta_{\bar{J}}^{(1)}$ is the limit of the sequence $\left(\frac{\zeta_{I,m}}{\zeta_{J,m}}\delta_{\bar{J},m}\right)_m$, and the divisor of the m-th element of the sequence is:

$$V_{\bar{I},*,m} - V_{\bar{J},*,m} + I - J + (d_J - d_I)\infty + \mathrm{Div}(\delta_{\bar{J},m}) = V_{\bar{I},*,m} + V_{\bar{J},m} + I - J - (2g + d_I - d_J)\infty.$$

By Proposition 3.2.9, $\frac{\zeta_I}{\zeta_J}\delta_{\bar{J}}^{(1)}$ is rational, with divisor:

$$\left(\lim_m (V_{\bar{I},*,m} + V_{\bar{J},m} + I + d_J\infty)\right) - J - (2g + d_I)\infty = V_{\bar{I},*}^{(1)} + V_{\bar{J}}^{(1)} + I - J - (2g + d_I - d_J)\infty.$$

In particular, $\frac{\zeta_I}{\zeta_J}$ is rational, with divisor:

$$V_{\bar{I},*}^{(1)} + V_{\bar{J}}^{(1)} + I - J - (2g + d_I - d_J)\infty - \mathrm{Div}(\delta_{\bar{J}}^{(1)}) = V_{\bar{I},*}^{(1)} - V_{\bar{J},*}^{(1)} + I - J + (d_J - d_I)\infty.$$

Since the positive and negative components of this divisor are H-rational and $\frac{\zeta_I}{\zeta_J}(\Xi) = \frac{\zeta_I(\Xi)}{\zeta_J(\Xi)} = 1$, $\frac{\zeta_I}{\zeta_J}$ is a rational function on X_H. Let's fix an $\mathbb{F}_q$-linear basis $(a_i)_i$ of A with increasing degree; if we have a sequence $(h_m = \sum_{i=0}^{k} c_{m,i} \otimes a_i)_m$ converging to $h = \sum_{i=0}^{k} c_i \otimes a_i$ in $A_{\mathbb{C}_\infty}$, and if the degree of h_m at ∞ is eventually equal to the degree of h at ∞ (i.e. $c_k \neq 0$), we have:

$$\lim_m \mathrm{sgn}(h_m) = \lim_m c_{m,k}\,\mathrm{sgn}(a_k) = c_k\,\mathrm{sgn}(a_k) = \mathrm{sgn}(h).$$

In particular, we deduce that $\lim_m \mathrm{sgn}(\delta_{\bar{J},m}) = \mathrm{sgn}(\delta_{\bar{J}})^q = 1$. If we fix any nonzero element $c \in J$, of degree d_c, the rational functions $\left(\frac{\zeta_{I,m}}{\zeta_{J,m}}\delta_{\bar{J},m}(1 \otimes c)\right)_m$ have only a pole at ∞, of degree $2g + d_I + (d_c - d_J)$, hence they belong to $A_{\mathbb{C}_\infty}$, and the same holds for their limit $\frac{\zeta_I}{\zeta_J}\delta_{\bar{J}}^{(1)}(1 \otimes c)$: we deduce that $\mathrm{sgn}\left(\frac{\zeta_I}{\zeta_J}\right) = \lim_m \mathrm{sgn}\left(\frac{\zeta_{I,m}}{\zeta_{J,m}}\right)$. Fix an $\mathbb{F}_q$-basis $(a_i)_i$ of I, with strictly increasing degrees and sign 1. We have:

$$\begin{aligned}\mathrm{sgn}(\zeta_{I,m}) = \sum_{a \in I(j_m)} a^{-1}\,\mathrm{sgn}(a) = -\sum_{a \in I(<j_m)} (a_{m+1} - a)^{-1} = -a_{m+1}^{-1} \sum_{i \geq 0} \sum_{a \in I(<j_m)} \frac{a^i}{a_{m+1}^i} \\ = -\, a_{m+1}^{-1} \sum_{i \geq 0} \sum_{e_1 + \cdots + e_m = i} \lambda(e_1, \ldots, e_m) \left(\frac{a_1}{a_{m+1}}\right)^{e_1} \cdots \left(\frac{a_m}{a_{m+1}}\right)^{e_m},\end{aligned}$$

where $\lambda(e_1, \ldots, e_m) \in \mathbb{F}_q$ is a certain coefficient. By Lemma 4.1.6, if $\lambda(e_1, \ldots, e_m) \neq 0$, we must have $e_j \geq q^j - q^{j-1}$ for $j = 1, \ldots, m$. In particular, since the norms of the elements $\left(\frac{a_j}{a_{m+1}}\right)_j$ are strictly increasing and less than 1, the unique summand of maximum norm corresponds to the m-uple $(e_j)_j = (q^j - q^{j-1})_j$. Since $\lambda((q^j - 1)_j) = \binom{q^m - 1}{q - 1, \ldots, q^m - q^{m-1}} = 1$, we get that the first term in the expansion of $\mathrm{sgn}(\zeta_{I,m})$ in $\mathbb{F}_q[[u]]$ is $-u^M$ for some integer M. Since the same argument holds for $\zeta_{J,m}$, we obtain that $\mathrm{sgn}\left(\frac{\zeta_I}{\zeta_J}\right) = u^N + o(u^N)$ for some integer N. □

Theorem 4.3.32. *Let ϕ be a normalized Drinfeld module of rank 1 with period lattice $\tilde{\pi}_I I$, where $\tilde{\pi}_I \in \mathbb{C}_\infty^\times$ and $I < A$ is a nonzero ideal. Fix an ideal $J < A$ of degree d_J such that $J\Omega \cong A$, and denote by h the unique rational function on X_H with* $\mathrm{sgn}(h) = 1$ *and* $\mathrm{Div}(h) = V_{\bar{I},*}^{(1)} + V_{\bar{I}+\bar{J}} - J - \Xi - (2g - d_J - 1)\infty$*. The following A-submodules of $\mathbb{C}_\infty \hat{\otimes} A$ coincide:*

$$\mathrm{Sf}_\phi(A) = \frac{(\tilde{\pi}_I \otimes 1)h}{\zeta_I}(\mathbb{F}_q \otimes IJ).$$

Proof. Let's denote by f the shtuka function associated to ϕ and f_* the adjoint shtuka function associated to to ϕ^* (recall Definition 4.3.12). By Remark 4.3.14,if we fix $a_I \in I$ of least degree, we have the equality $a_I \tilde{\pi}_I = \tilde{\pi}_\phi$ up to a factor in $\mathbb{F}_q^\times$, hence by Proposition 4.3.24 we have $\frac{\left((\tilde{\pi}_I^{-1} \otimes 1)\zeta_I\right)^{(-1)}}{(\tilde{\pi}_I^{-1} \otimes 1)\zeta_I} = f_*$. On the other hand, by Remark 4.2.3 and Proposition 4.3.23 respectively, we have the identities $\mathrm{Div}(f) = V_{\bar{I}+\bar{J}}^{(1)} - V_{\bar{I}+\bar{J}} + \Xi - \infty$ and $\mathrm{Div}(f_*) = V_{\bar{I},*} - V_{\bar{I},*}^{(1)} + \Xi - \infty$, hence:

$$\mathrm{Div}\left(\frac{h^{(1)}}{h}\right) = V_{\bar{I},*}^{(2)} + V_{\bar{I}+\bar{J}}^{(1)} - \Xi^{(1)} - V_{\bar{I},*}^{(1)} - V_{\bar{I}+\bar{J}} + \Xi = \mathrm{Div}\left(\frac{f}{f_*^{(1)}}\right);$$

since the rational functions $\frac{h^{(1)}}{h}$ and $\frac{f}{f_*^{(1)}}$ both have sign 1, they coincide. In particular, we have that:

$$\left(\frac{(\tilde{\pi}_I \otimes 1)h}{\zeta_I}\right)^{(1)} \left(\frac{(\tilde{\pi}_I \otimes 1)h}{\zeta_I}\right)^{-1} = \frac{h^{(1)}}{h} f_*^{(1)} = f,$$

hence $\frac{(\tilde{\pi}_I \otimes 1)h}{\zeta_I} \in (\mathbb{F}_q \otimes K)\,\mathrm{Sf}_\phi(A)$. By Theorem 4.2.1, the A-module $\mathrm{Sf}_\phi(A) \subseteq \mathbb{C}_\infty \hat{\otimes} A$ coincides with $(\mathbb{F}_q \otimes IJ)\frac{\delta_{\bar{I}+\bar{J}}(\lambda \otimes 1)}{f_{\bar{I}+\bar{J},*}\zeta_{IJ}}$, where $\lambda \in \mathbb{C}_\infty$ is some nonzero constant. To conclude the proof we just need to show that the product $\left(\frac{\delta_{\bar{I}+\bar{J}}}{f_{\bar{I}+\bar{J},*}\zeta_{IJ}}\right)\left(\frac{h}{\zeta_I}\right)^{-1}$ is a constant, i.e. it is a rational function with trivial divisor. By Proposition 4.3.31, $\frac{\zeta_{IJ}}{\zeta_I}$ is a rational function, and we get the following:

$$\begin{aligned}
&\mathrm{Div}\left(\frac{\delta_{\bar{I}+\bar{J}}}{f_{\bar{I}+\bar{J},*}\zeta_{IJ}}\frac{\zeta_I}{h}\right) = \mathrm{Div}(\delta_{\bar{I}+\bar{J}}) - \mathrm{Div}(f_{\bar{I}+\bar{J},*}) + \mathrm{Div}\left(\frac{\zeta_I}{\zeta_{IJ}}\right) - \mathrm{Div}(h)\\
=&(V_{\bar{I}+\bar{J}} + V_{\bar{I}+\bar{J},*} - 2g\infty) + (-V_{\bar{I}+\bar{J},*} + V_{\bar{I}+\bar{J},*}^{(1)} - \Xi + \infty) +\\
+&(V_{\bar{I},*}^{(1)} - V_{\bar{I}+\bar{J},*}^{(1)} - J + d_J\infty) + (-V_{\bar{I},*}^{(1)} - V_{\bar{I}+\bar{J}} + J + \Xi + (2g - d_J - 1)\infty) = 0.
\end{aligned}$$

□

In the notation of the previous theorem and of Lemma 4.3.30 we give the next definition.

Definition 4.3.33. Fix an element $a_I \in I$ of least degree. We define the *pseudocanonical special function* as follows:

$$\omega_{\phi,J} := \frac{(\pi_\phi \otimes 1)h_{J,\bar{I}}}{\zeta_I}(a_I^{-1} \otimes a_I) \in \mathbb{C}_\infty \hat{\otimes} K.$$

Remark 4.3.34. Since $\tilde{\pi}_\phi = \tilde{\pi}_I a_I$ up to a factor in $\mathbb{F}_q^\times$, by Theorem 4.3.32 $\omega_{\phi,J}$ belongs to $\mathrm{Sf}_\phi(A)(\mathbb{F}_q \otimes K)$, and is well defined up to a factor in $\mathbb{F}_q^\times$. While $\omega_{\phi,J}$ depends on the choice of the ideal J, it does not depend on the choice of I but only on its class.

4.4 An identity involving special functions and zeta functions à la Anderson

Fix a normalized Drinfeld module of rank 1 $\phi : A \to H[\tau]$ with period lattice $\tilde{\pi}_I I$, where $I < A$ is a nonzero ideal – so that its Drinfeld divisor is $V_{\bar{I}-\bar{\Omega}}$ by Remark 4.2.3, where $\bar{\Omega} \in Cl(A)$ denotes the isomorphism class of Ω – and call $f = f_{\bar{I}-\bar{\Omega}}$ its shtuka function. Following a construction of Hayes (see [Hay79]), we can define ϕ_J for any nonzero ideal $J = (a,b) < A$ as the unique monic generator ϕ_J of the left ideal $(\phi_a, \phi_b) < H[\tau]$.

Definition 4.4.1. Fix a nonzero $\omega \in \mathrm{Sf}_\phi(A)$, and for all nonzero ideals $J < A$ define:

$$\chi_\phi(J) := \frac{\phi_J(\omega)}{\omega}.$$

Remark 4.4.2. The previous definition does not depend on the choice of ω.

In his paper [And94], Anderson proved a famous log-algebraicity result.

Theorem 4.4.3 ([And94][Thm. 5.1.1])**.** *Let B be the integral closure of A in H. For all $b \in B$, the power series*

$$\exp_\phi\left(\sum_{\sigma \in \mathrm{Gal}\left(H/K\right)} \sum_{J<A} \frac{b^\sigma}{\chi_\phi(J)(\Xi)} z^{q^{\deg(I)}}\right) \in H[[z]]$$

is contained in $B[z]$.

This result was subsequently generalized by Anderson himself, who added a parameter in the variable b ([And96][Thm. 3]). Anglès, Ngo Dac and Tavares Ribeiro used the module of Stark units to improve on Anderson's results ([ANT17b][Thm. 4.2]), and further developed this line of research in [ANT22], where they proved Taelman's class formula for arbitrary Drinfeld modules.

In their paper [GP18], Green and Papanikolas followed another approach. They introduced a zeta function "à la Anderson" $\xi_\phi \in \mathbb{C}_\infty \hat{\otimes} A$ and gave an explicit expression of its product with a special function ω when X is an elliptic curve ([GP18, Thm. 7.3]), proving that it is a rational function on X_H (a similar rationality result for arbitrary genera, without an explicit expression but without are contained in [ANT17a]). Green and Papanikolas then went on to use similar techniques to give a different proof of Anderson's theorem [And94][Thm. 5.1.1] in [GP18][Thm. 8.1].

In this section, we prove the generalization of [GP18, Thm. 7.3] to a curve X of arbitrary genus in the form of Theorem 4.4.11, where we relate ξ_ϕ to a pseudocanonical special function.

Remark 4.4.4. Fix a nonzero $a \in A$ of sign 1. We have:

$$\chi_\phi((a)) = \frac{\phi_{(a)}(\omega)}{\omega} = \frac{((1 \otimes a)\omega)}{\omega} = 1 \otimes a.$$

Moreover, for any nonzero ideal $J < A$, since $\phi_{aJ} = \phi_J \circ \phi_a$ we have that

$$\chi_\phi(aJ) = \frac{\phi_{aJ}(\omega)}{\omega} = \frac{\phi_J \circ \phi_a(\omega)}{\omega} = \frac{\phi_J((1 \otimes a)\omega)}{\omega} = (1 \otimes a)\frac{\phi_J(\omega)}{\omega} = \chi_\phi((a))\chi_\phi(J).$$

It's easy to check that we can extend χ_ϕ to all fractional ideals in a unique way such that for all $c \in K$ and for all fractional ideals J we have $\chi_\phi((c))\chi_\phi(J) = \chi_\phi(cJ)$.

Proposition 4.4.5. *For all nonzero ideals $J < A$, $\chi_\phi(J)$ is a rational function on X_H with sign 1, and* $\mathrm{Div}(\chi_\phi(J)) = V_{\bar{I}-\bar{\Omega}-\bar{J}} + J - V_{\bar{I}-\bar{\Omega}} - \deg(J)\infty$.

Proof. Consider $H^0(X_H \setminus \{\infty\}, V_{\bar{I}-\bar{\Omega}}) = \bigcup_{k\geq 0} H^0(X_H, V_{\bar{I}-\bar{\Omega}} + k\infty)$, which admits $\{f \cdots f^{(k-1)}\}_{k\geq 0}$ as a basis. For a fixed non principal ideal $J = (a,b) < A$, if we write $\phi_J = \sum_{i=0}^{\deg(J)} (c_i \otimes 1)\tau^i$, we get:

$$\chi_\phi(J) = \frac{\phi_J(\omega)}{\omega} = \sum_{i=0}^{\deg(J)} c_i f \cdots f^{(i-1)} \in H^0(X_H, V_{\bar{I}-\bar{\Omega}} + \deg(J)\infty).$$

Since $c_{\deg(J)} f \cdots f^{(\deg(J)-1)}$ is the summand with the pole of highest degree at ∞, and since $c_{\deg(J)} = 1$ and $\mathrm{sgn}(f) = 1$, we have:

$$\mathrm{sgn}(\chi_\phi(J)) = c_{\deg(J)} \,\mathrm{sgn}(f) \cdots \mathrm{sgn}(f)^{(\deg(J)-1)} = 1.$$

Moreover, if we write $\phi_J = \psi_1 \circ \phi_a + \psi_2 \circ \phi_b$ for some $\psi_1, \psi_2 \in H[\tau]$, we get:

$$\chi_\phi(J) = \frac{\phi_J(\omega)}{\omega} = \frac{\psi_1 \circ \phi_a(\omega) + \psi_2 \circ \phi_b(\omega)}{\omega} = (1 \otimes a)\frac{\psi_1(\omega)}{\omega} + (1 \otimes b)\frac{\psi_2(\omega)}{\omega}.$$

Since $1 \otimes a, 1 \otimes b \in H^0(X_H \setminus \{\infty\}, -J)$, $\frac{\psi_1(\omega)}{\omega}, \frac{\psi_2(\omega)}{\omega} \in H^0(X_H \setminus \{\infty\}, V_{\bar{I}-\bar{\Omega}})$, and the degree of $\chi_\phi(J)$ is $\deg(J)$, we get $\chi_\phi(J) \in H^0(X_H, V_{\bar{I}-\bar{\Omega}} - J + \deg(J)\infty)$. The divisor $D := V_{\bar{I}-\bar{\Omega}} + \deg(J)\infty - J$ has degree g and is such that:

$$[D - D^{(1)}] = [V_{\bar{I}-\bar{\Omega}} - V_{\bar{I}-\bar{\Omega}}^{(1)}] = [\Xi - \infty] \qquad\qquad \mathrm{red}([D - g\infty]) = \bar{I} - \bar{\Omega} - \bar{J}.$$

By Proposition 3.3.25, $D \sim V_{\bar{I}-\bar{\Omega}-\bar{J}}$, and $h^0(V_{\bar{I}-\bar{\Omega}-\bar{J}}) = 1$, hence the divisor $\mathrm{Div}(\chi_\phi(J))$ is equal to $V_{\bar{I}-\bar{\Omega}-\bar{J}} + J - V_{\bar{I}-\bar{\Omega}} - \deg(J)\infty$. □

Let's include an easy Lemma.

Lemma 4.4.6. *For any nonzero ideal $J < A$, as a $\mathbb{F}_q$-linear endomorphism of $\mathbb{C}_\infty$,* $\ker(\phi_J) = \exp_\phi(\tilde{\pi}_I I J^{-1})$.

Proof. We can fix two generators a, b of J. By definition of ϕ_J, there are ψ_1, ψ_2 in $H[\tau]$ such that $\phi_J = \psi_1 \circ \phi_a + \psi_2 \circ \phi_b$. Since for any $x \in \tilde{\pi}_I I J^{-1}$, ax and bx belong to $\tilde{\pi}_I I = \ker(\exp_\phi)$, we have:

$$\phi_J \circ \exp_\phi(x) = \psi_1 \circ \phi_a \circ \exp_\phi(x) + \psi_2 \circ \phi_b \circ \exp_\phi(x) = \psi_1 \circ \exp_\phi(ax) + \psi_2 \circ \exp_\phi(bx) = 0.$$

In particular, $\exp_\phi(\tilde{\pi}_I I J^{-1}) \subseteq \ker(\phi_J)$. On the other hand, $\exp_\phi(\tilde{\pi}_I I J^{-1})$ is isomorphic as an $\mathbb{F}_q$-vector space to ${}^{\tilde{\pi}_I I J^{-1}}\!/\!_{\tilde{\pi}_I I}$, which has cardinality $q^{\deg(J)}$. Since ϕ_J is a polynomial of degree $q^{\deg(J)}$, we get the equality $\exp_\phi(\tilde{\pi}_I I J^{-1}) = \ker(\phi_J)$. □

Proposition 4.4.7. *For all nonzero ideals $J < A$,* $\mathrm{red}_u(\chi_\phi(J)(\Xi)) = 1$.

Proof. If $J = (a)$, where $a \in A \setminus \{0\}$ has sign 1, $\mathrm{red}_u(\chi_\phi(J)(\Xi)) = \mathrm{red}_u(a \otimes 1) = 1$. In particular, for any nonzero ideal J and for all $a \in A \setminus \{0\}$, $\mathrm{red}_u(\chi_\phi(J)(\Xi))$ is equal to $\mathrm{red}_u(\chi_\phi(aJ)(\Xi))$. If we write $\phi_J = \sum_{i=0}^{\deg(J)} c_i \tau^i$, we have:

$$c_0 = \left(\sum_{i=0}^{\deg(J)} (c_i \otimes 1) f \cdots f^{(i-1)}\right)(\Xi) = \chi_\phi(J)(\Xi),$$
$$c_{\deg(J)} = 1.$$

By Lemma 4.4.6, $\ker(\phi_J) = \exp_\phi(\tilde{\pi}_I IJ^{-1})$; let's fix a set $\{\alpha_i\}_i \subseteq IJ^{-1}$ such that the elements $\{\exp_\phi(\tilde{\pi}_I\alpha_i)\}_i$ are representatives for the quotient ${(\ker(\phi_J) \setminus \{0\})}/{\mathbb{F}_q^\times}$. We have:

$$\chi_\phi(J)(\Xi) = c_0 = \prod_{\beta\in\ker(\phi_J)\setminus\{0\}} \beta = \prod_{\lambda\in\mathbb{F}_q^\times}\prod_i \lambda\exp_\phi(\tilde{\pi}_I\alpha_i) = \prod_i -(\exp_\phi(\tilde{\pi}_I\alpha_i))^{q-1}.$$

Let's write $\exp_\phi = \sum_{k\geq 0} e_k\tau^k$; by Remark 4.3.16 $e_k \in H \subseteq K_\infty$ for all k. In particular, for all i the element $\gamma_i := \frac{\exp_\phi(\tilde{\pi}_I\alpha_i)}{\tilde{\pi}_I} = \alpha_i\sum_j e_j(\tilde{\pi}_I\alpha_i)^{q^j-1}$ is contained in K_∞, which means that $\mathrm{red}_u(\gamma_i^{q-1}) = 1$. Since the cardinality of $\{\alpha_i\}_i$ is $\frac{q^{\deg(J)}-1}{q-1}$, we have:

$$\begin{aligned}\mathrm{red}_u\left(\chi_\phi(J)(\Xi)\right) &= \prod_i -\mathrm{red}_u\left(\tilde{\pi}_I^{q-1}\right)\mathrm{red}_u(\gamma_i)^{q-1} = \mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right)^{\frac{q^{\deg(J)}-1}{q-1}}\\ &= \mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right)^{\deg(J)}.\end{aligned}$$

For $d \gg 0$, we can pick $a, b \in A\setminus\{0\}$ with $\deg(a) = \deg(b) - 1$. We have:

$$\begin{aligned}1 &= \frac{\mathrm{red}_u\left(\chi_\phi(bJ)(\Xi)\right)}{\mathrm{red}_u\left(\chi_\phi(aJ)(\Xi)\right)} = \frac{\mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right)^{\deg(bJ)}}{\mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right)^{\deg(aJ)}} = \mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right)^{\deg(b)-\deg(a)}\\ &= \mathrm{red}_u\left(-\tilde{\pi}_I^{q-1}\right),\end{aligned}$$

hence $\mathrm{red}_u\left(\chi_\phi(J)(\Xi)\right) = 1$. □

Remark 4.4.8. Following a known construction due to Hayes ([Hay79][Section 3]), for any nonzero ideal $J < A$ we denote by $\phi^J : A \to H[\tau]$ the unique ring homomorphism such that for all $a \in A$ $\phi^J_a \circ \phi_J = \phi_J \circ \phi_a$. It can be easily shown (see e.g. [Gos98][Subsection 4.9]) that the morphism $\phi^J : A \to H[\tau]$ is a normalized Drinfeld module of rank 1 and its associated lattice is $\chi_\phi(J)(\Xi)\tilde{\pi}_I J^{-1}I$.

Definition 4.4.9. Call $a_I \in I$ the unique nonzero element of least degree with sign 1. The *Anderson zeta function* relative to the Drinfeld module ϕ is defined as:

$$\xi_\phi := \sum_{J<a_I^{-1}I} \frac{\chi_\phi(J)}{\chi_\phi(J)(\Xi)} \in K_\infty\hat{\otimes}K.$$

Remark 4.4.10. In the previous definition, ξ_ϕ only depends on the ideal class $\bar{I}$ of I. We can also write $\xi_\phi = a_I \otimes a_I^{-1}\sum_{J<I}\frac{\chi_\phi(J)}{\chi_\phi(J)(\Xi)}$.

The series ξ_ϕ was defined in the special case of $g(X) = 1$ and $I = A$ by Green and Papanikolas ([GP18][Eq. (95)]), who proved that it is a rational multiple of $\frac{\tilde{\pi}_\phi\otimes 1}{\omega_\phi}$ and gave an explicit formula for their quotient ([GP18][Thm. 7.3]). At the same time, a similar series was studied by Anglès, Ngo Dac, and Tavares Ribeiro, who proved a similar statement for general A, but without expressing the proportionality factor ([ANT17a]).

In the notation of Lemma 4.3.30, we finally state and prove the main theorem of this section, a direct generalization of [GP18][Thm. 7.3].

Theorem 4.4.11. *The Anderson zeta function ξ_ϕ is a well defined element of $K_\infty \hat{\otimes} K$. Moreover, if we fix an ideal $J < A$ such that $J\Omega \cong A$, the following identity holds:*

$$\xi_\phi \omega_{\phi,J} = (\tilde{\pi}_\phi \otimes 1) \sum_{\sigma \in \mathrm{Gal}(H/K)} h^\sigma_{J,\bar{A}}.$$

Proof. Let's fix representatives $J_i < I$ for each ideal class $\bar{J}_i \in Cl(A)$. To prove convergence we rearrange the terms:

$$\sum_{\tilde{J}<I} \frac{\chi_\phi(\tilde{J})}{\chi_\phi(\tilde{J})(\Xi)} = \sum_i \sum_{\substack{\tilde{J}<I \\ \tilde{J}\cong J_i}} \frac{\chi_\phi(\tilde{J})}{\chi_\phi(\tilde{J})(\Xi)} = \sum_i \sum_{\substack{a \in J_i^{-1} I \setminus \{0\} \\ \mathrm{sgn}(a)=1}} \frac{\chi_\phi(aJ_i)}{\chi_\phi(aJ_i)(\Xi)}$$

$$= -\sum_i \sum_{a \in J_i^{-1} I \setminus \{0\}} \frac{\chi_\phi(aJ_i)}{\chi_\phi(aJ_i)(\Xi)} = -\sum_i \left(\frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)} \sum_{a \in J_i^{-1} I \setminus \{0\}} \frac{\chi_\phi(a)}{\chi_\phi(a)(\Xi)} \right)$$

$$= \sum_i \frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)} \zeta_{J_i^{-1} I}.$$

Since $\omega_{\phi,J}$ is defined as $\frac{(\tilde{\pi}_\phi \otimes 1) h_{J,\bar{I}}}{\zeta_I}(a_I^{-1} \otimes a_I)$, we get:

$$\xi_\phi \omega_{\phi,J} = (\tilde{\pi}_\phi \otimes 1) \sum_i \frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)} \frac{\zeta_{J_i^{-1} I}}{\zeta_I} h_{J,\bar{I}}.$$

For all i, by Proposition 4.3.31 $\frac{\zeta_{J_i^{-1}I}}{\zeta_I}$ is a rational function on X_H; the evaluation at Ξ of the rational function $\frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)}$ is 1, and by Proposition 4.4.5 and Remark 3.3.26 the positive and negative components of its divisor is H-rational, hence the function is defined on X_H; finally, $h_{J_i,\bar{I}}$ is a rational function on X_H by Lemma 4.3.30. We deduce that for all i the summand $\frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)} \frac{\zeta_{J_i^{-1}I}}{\zeta_I} h_{J,\bar{I}}$ is a rational function on X_H, and by Proposition 4.4.5 and Lemma 4.3.30 its divisor is:

$$\begin{aligned}
&(V^{(1)}_{\bar{I}-\bar{J}_i,*} - V^{(1)}_{\bar{I},*} - J_i) + (V_{\bar{I}+\bar{J}-\bar{J}_i} + J_i - V_{\bar{I}+\bar{J}}) + \\
&+(V^{(1)}_{\bar{I},*} + V_{\bar{I}+\bar{J}} - J - \Xi) - (2g - \deg(J) - 1)\infty \\
&= V^{(1)}_{\bar{I}-\bar{J}_i,*} + V_{\bar{I}+\bar{J}-\bar{J}_i} - J - \Xi - (2g - \deg(J) - 1)\infty = \mathrm{Div}(h_{J,\bar{I}-\bar{J}_i}),
\end{aligned}$$

hence $\frac{\zeta_{J_i^{-1}I}}{\zeta_I} \frac{\chi_\phi(J_i)}{\chi_\phi(J_i)(\Xi)} h_{J,\bar{I}} = (\alpha_i \otimes 1) h_{J,\bar{I}-\bar{J}_i}$ for some $\alpha_i \in H^\times$. Since $\mathrm{sgn}(h_{J,\bar{I}-\bar{J}_i}) = 1$, α_i is equal to the sign of the summand. For all i denote $\tilde{\pi}_{J_i^{-1}I} := \chi_\phi(J)(\Xi)\tilde{\pi}_I$: By Remark 4.4.8, if ψ is the unique normalized Drinfeld module of rank 1 whose period lattice Λ_ψ is isomorphic to $J_i^{-1}I$, then $\Lambda_\psi = \tilde{\pi}_{J_i^{-1}I} J_i^{-1} I$. If we denote $s_i := \chi_\phi(J_i)(\Xi)^{-1} \zeta_I^{-1} \zeta_{J_i^{-1}I}$, by Proposition 4.3.24 we have:

$$\frac{s_i^{(1)}}{s_i} = \frac{\left(\chi_\phi(J_i)(\Xi)^{-1}\zeta_{J_i^{-1}I}\right)^{(1)}}{\chi_\phi(J_i)(\Xi)^{-1}\zeta_{J_i^{-1}I}} \cdot \frac{\zeta_I}{\zeta_I^{(1)}} = \frac{\left(\tilde{\pi}^{-1}_{J_i^{-1}I}\zeta_{J_i^{-1}I}\right)^{(1)}}{\tilde{\pi}^{-1}_{J_i^{-1}I}\zeta_{J_i^{-1}I}} \cdot \frac{\tilde{\pi}_I^{-1}\zeta_I}{\left(\tilde{\pi}_I^{-1}\zeta_I\right)^{(1)}} = \left(\frac{f_{\bar{I},*}}{f_{\bar{I}-\bar{J}_i,*}} \right)^{(1)}$$

$$\Rightarrow \mathrm{sgn}(s_i)^{q-1} = \mathrm{sgn}\left(\frac{s_i^{(1)}}{s_i}\right) = \frac{\mathrm{sgn}(f_{\bar{I},*})^q}{\mathrm{sgn}(f_{\bar{I}-\bar{J},*})^q} = 1;$$

in particular,

$$\alpha_i = \mathrm{sgn}(s_i) \cdot \mathrm{sgn}(\chi_\phi(J_i)) \cdot \mathrm{sgn}(h_{J,\bar{I}}) = \mathrm{sgn}(s_i) \in \mathbb{F}_q.$$

On the other hand, by Proposition 4.3.31 and Proposition 4.4.7 we have:

$$\mathrm{red}_u(\mathrm{sgn}(s_i)) = \mathrm{red}_u(\mathrm{sgn}(\chi_\phi(J_i)(\Xi)))^{-1} \cdot \mathrm{red}_u\left(\mathrm{sgn}\left(\frac{\zeta_{J_i^{-1}I}}{\zeta_I}\right)\right) = 1,$$

hence $\alpha_i = 1$. We can rewrite:

$$\xi_\phi \omega_{\phi,J} = (\tilde{\pi}_\phi \otimes 1) \sum_i h_{J,\bar{I}-\bar{J}_i} = (\tilde{\pi}_\phi \otimes 1) \sum_{\sigma \in \mathrm{Gal}(H/K)} h^\sigma_{J,\bar{A}}. \qquad \square$$

Remark 4.4.12. Evaluating at Ξ we get $\xi_{\bar{I}}(\Xi) = \#Cl(A)$ modulo the characteristic of $\mathbb{F}_q$, generalizing [GP18][Corollary 7.4].

Chapter 5

Pellarin zeta functions and Pellarin's identity for arbitrary Drinfeld A-modules

As in the previous chapters, we let ${X}/{\mathbb{F}_q}$ be a smooth, projective, geometrically irreducible curve of genus $g(X)$ with a closed point $\infty \in X$. We drop the assumption that ∞ is $\mathbb{F}_q$-rational, and denote by e its degree. Recall that A denotes the ring $H^0(X \setminus \{\infty\}, \mathcal{O}_X)$ of rational functions with only poles at ∞, K_∞ denotes the completion of K at ∞, and $\mathbb{C}_\infty$ denotes the completion of an algebraic closure of K_∞.

In Definition 2.2.7, we introduced the functor of Anderson eigenvectors relative to an arbitrary Anderson module $\underline{E}$. Replacing the action of ϕ by its adjoint ϕ^*, in Definition 5.2.8 we introduce the functor of *dual Anderson eigenvectors* $\mathrm{Sf}_{\phi^*} : A - \mathbf{Mod} \to A - \mathbf{Mod}$. In the case of a Drinfeld module $\underline{E} = (\mathbb{G}_a, \phi)$ of rank 1 with period lattice Λ_ϕ, we have proven in Theorem 4.3.27 that the series $\zeta := -\sum_{\lambda \in \Lambda_\phi} \lambda^{-1} \otimes \lambda \in \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ satisfies the following identity for all $a \in A$:

$$(\phi_a^* \otimes 1)\zeta = (1 \otimes a)\zeta.$$

Using the functorial language of Definition 5.2.8, we generalize Theorem 4.3.27 to arbitrary Drinfeld modules with a result analogous to Theorem 2.2.9.

Theorem (Thm. 5.2.10)**.** *Let $(\mathbb{G}_a, \phi)$ be a Drinfeld module of rank r. The functor Sf_{ϕ^*} is naturally isomorphic to $\mathrm{Hom}_A(\Lambda_\phi, \_)$; moreover, the universal object in $\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ corresponds to the map $\hat{\Lambda}_\phi \cong \ker \exp_\phi^* \subseteq \mathbb{C}_\infty$ and can be expressed as $-\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda$.*

We call the universal object $\zeta_\phi \in \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ the *universal dual Anderson eigenvector*, in analogy with the universal Anderson eigenvector $\omega_\phi \in \mathbb{C}_\infty \hat{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega)$.

As said in the introduction, we define a natural $\mathbb{C}_\infty \overline{\otimes} A$-bilinear pairing

$$\_ \cdot \_ : \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi \times \mathbb{C}_\infty \hat{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega) \to \mathbb{C}_\infty \hat{\otimes} \Omega \cong \mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({K_\infty}/{A}, \mathbb{C}_\infty\right),$$

and evaluate the pair $(\zeta_\phi, \omega_\phi)$. If $r = 1$, we can immerse both spaces on the right hand side in $\mathbb{C}_\infty \hat{\otimes} A$ (although in a noncanonical way), and this pairing can be identified with the usual product of elements in the algebra $\mathbb{C}_\infty \hat{\otimes} A$. With this in mind, the following result can be interpreted as a partial generalization of Theorem 4.3.32 to Drinfeld modules of arbitrary rank.

Theorem (Thm. 5.4.2)**.** *Let ϕ be a Drinfeld module with period lattice Λ_ϕ, and denote by ω_ϕ and ζ_ϕ the universal objects of the functors Sf_ϕ and Sf_{ϕ^*}, respectively. For all integers k, the pairing $\zeta_\phi \cdot \left(\tau^k\omega_\phi\right)$ in $\mathbb{C}_\infty \overline{\otimes}\Omega$ is a rational differential form on $X_{\mathbb{C}_\infty}$.*

5.1 Some remarkable identities in $\mathbb{C}_\infty$

5.1.1 Poonen duality

Definition 5.1.1. Let $\mathbb{C}_\infty[[\tau,\tau^{-1}]]$ denote the set of bilateral formal power series in τ. For any formal series $s = \sum_i s_i\tau^i \in \mathbb{C}_\infty[[\tau,\tau^{-1}]]$, we define its adjoint as $s^* := \sum_i \tau^{-i}s_i = \sum_i s_i^{q^{-i}}\tau^{-i} \in \mathbb{C}_\infty[[\tau,\tau^{-1}]]$.

Remark 5.1.2. Multiplication by elements of the noncommutative ring $\mathbb{C}_\infty[\tau,\tau^{-1}]$ endows the set $\mathbb{C}_\infty[[\tau,\tau^{-1}]]$ with a $\mathbb{C}_\infty[\tau,\tau^{-1}]$-bimodule structure.

Denote by $\exp_\phi \in \mathbb{C}_\infty[[\tau]] \subseteq \mathbb{C}_\infty[[\tau,\tau^{-1}]]$ the exponential relative to ϕ.

Remark 5.1.3. With similar arguments as in Remark 4.3.19, it's easy to show that, since $\exp_\phi$ has an infinite radius of convergence, the adjoint exponential $\exp_\phi^* \in \mathbb{C}_\infty[[\tau^{-1}]] \subseteq \mathbb{C}_\infty[[\tau,\tau^{-1}]]$ also converges everywhere on $\mathbb{C}_\infty$ (see also [Poon96][Prop. 1]).

We follow a construction due to Poonen, who proved a duality result of central importance to this section ([Poon96][Thm. 10]).

Lemma 5.1.4. *For all $\beta \in \ker(\exp_\phi^*) \setminus \{0\}$, there is a unique element $g_\beta \in \mathbb{C}_\infty[[\tau]]$ such that $(1-\tau)g_\beta = \beta\exp_\phi$. Moreover, g_β has infinite radius of convergence.*

Proof. Let's consider the following formal series:

$$h := \sum_{i\geq 0}\tau^i \in \mathbb{C}_\infty[[\tau]];$$

since $h(1-\tau)=1$, the defining property of g_β is equivalent to the identity $g_\beta = h\beta\exp_\phi$: this proves both existence and uniqueness. If we call e_i the i-th coefficient of $\exp_\phi$ and c_i the i-th coefficient of g_β, from the identity $g_\beta = h\beta\exp_\phi$ we get the following:

$$c_k = \sum_{i=0}^k \beta^{q^i} e_{k-i}^{q^i}\ \forall k \in \mathbb{Z}_{\geq 0} \Longrightarrow \lim_k c_k^{\frac{1}{q^k}} = \lim_k \sum_{i=0}^k e_i^{q^{-i}}\beta^{q^{-i}} = \exp_\phi^*(\beta) = 0,$$

hence the radius of convergence of g_β is infinite. □

Remark 5.1.5. Since $\ker((1-\tau)g_\beta) = \ker(\beta\exp_\phi) = \Lambda_\phi$, $g_\beta|_{\Lambda_\phi}$ has image in $\mathbb{F}_q$.

By convention, we set $g_0 = 0$. Recall that $\hat{\Lambda}_\phi = \mathrm{Hom}_{\mathbb{F}_q}(\Lambda_\phi,\mathbb{F}_q)$ has a natural topology which makes it a compact A-module. We state [Poon96][Thm. 10] in the special case of the exponential function.

Theorem 5.1.6. *The function $\psi : \ker(\exp_\phi^*) \to \hat{\Lambda}_\phi$ sending β to $g_\beta|_{\Lambda_\phi}$ is an A-linear homeomorphism, where $a \in A$ acts on the left hand side by sending $c \in \ker(\exp_\phi^*)$ to $\phi_a^*(c)$.*

The following proposition, which is proven in Subsection 5.1, can be viewed as an explicit algebraic formula for the inverse of the isomorphism in Theorem 5.1.6.

Proposition (Proposition 5.1.18)**.** *For all $\beta \in \ker(\exp_\phi^*) \setminus \{0\}$, the following identity holds in $\mathbb{C}_\infty$:*

$$\beta = -\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{g_\beta(\lambda)}{\lambda}.$$

In the following subsection, we include some technical lemmas necessary for the proof of Proposition 5.1.18. With the same lemmas we are also able to prove the following proposition, where we denote by $\log_\phi$ the inverse of the exponential $\exp_\phi$ in $\mathbb{C}_\infty[[\tau]]$; we also denote by l_i and e_i the i-th coefficient of $\log_\phi$ and $\exp_\phi$, respectively, and we set by convention $l_i = e_i = 0$ for all $i < 0$.

Proposition (Proposition 5.1.19)**.** *Let ϕ be a Drinfeld module of rank r. For all integers k, for all $c \in K_\infty \setminus \{0\}$ with $\|c\| \le q^{\frac{k-1}{r}}$, the following identity holds in $\mathbb{C}_\infty$:*

$$\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{\exp_\phi(c\lambda)}{\lambda^{q^k}} = -\sum_{i+j=k} e_j l_i^{q^j} c^{q^j}.$$

5.1.2 Lattices

Throughout this subsection, $\mathbf{C}$ is a complete normed K_∞-vector space (with nonarchimedean norm).

Definition 5.1.7. An infinite $\mathbb{F}_q$-linear subspace $V \subseteq \mathbf{C}$ is a *lattice* if for any positive real number r there are finitely many elements of V of norm at most r.

An *ordered basis* of V is a sequence $(v_i)_{i\ge 1}$ with the following property: for all $m \ge 1$, v_m is an element of $V \setminus \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<m})$ of least norm. If such a basis exists, we call the sequence of real numbers $(\|v_i\|)_{i\ge 1}$ a *norm sequence* of V.

Remark 5.1.8. If $V \subseteq \mathbf{C}$ is a lattice, every subset $S \subseteq V$ has an element of least norm. In particular, since V is infinite, we can construct an ordered basis of V by recursion.

The next two results justify the terminology "ordered basis" and prove that the norm sequence exists and is unique.

Lemma 5.1.9. *If $(v_i)_{i\ge 1}$ is an ordered basis of a lattice $V \subseteq \mathbf{C}$, it is a basis of V as an $\mathbb{F}_q$-vector space.*

Proof. For all $m \ge 1$ $v_m \notin \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<m})$, hence the v_i's are $\mathbb{F}_q$-linearly independent. Since for all $r \in \mathbb{R}$ there is a finite number of elements of V with norm at most r, the norm sequence $(\|v_i\|)_{i\ge 1}$ tends to infinity; in particular, for all $v \in V$ there is an integer m such that $\|v_m\| > \|v\|$, so $v \in \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<m})$ by construction of v_m. □

Proposition 5.1.10. *If $(v_i)_{i\ge 1}$ is an ordered basis of a lattice $V \subseteq \mathbf{C}$, and $(v_i')_{i\ge 1}$ is a sequence of elements in V that are $\mathbb{F}_q$-linearly independent and whose norms form a weakly increasing sequence of real numbers, then $\|v_i'\| \ge \|v_i\|$ for all i. In particular, the norm sequence of V does not depend on the chosen ordered basis of V.*

Proof. By contradiction, suppose $\|v_m'\| < \|v_m\|$ for some m. Then for all $i \le m$ we have $\|v_i'\| \le \|v_m'\| < \|v_m\|$, so $v_i' \in \mathrm{Span}_{\mathbb{F}_q}(\{v_j\}_{j<m})$ by construction of v_m; since $\{v_i'\}_{i\le m}$ is a set of m $\mathbb{F}_q$-linearly independent vectors and $\dim_{\mathbb{F}_q} \mathrm{Span}_{\mathbb{F}_q}(\{v_j\}_{j<m}) = m-1$, we have reached a contradiction. If we take $(v_i')_i$ to be another ordered basis, by this reasoning we get both $\|v_m'\| \ge \|v_m\|$ and $\|v_m\| \ge \|v_m'\|$, hence the norm sequence is independent from the choice of the ordered basis. □

Finally, we show that the norm sequence of a subspace is reasonably well behaved.

Lemma 5.1.11. *Let $W \subseteq V \subseteq \mathbf{C}$ be lattices. The norm sequence $(s_i)_{i\geq 1}$ of W is a subsequence of the norm sequence $(r_i)_{i\geq 1}$ V.*

Moreover, if $\dim_{\mathbb{F}_q} {}^{V}\!/_{W} = n < \infty$, for $i \gg 0$ we have $r_i = s_{i+n}$.

Proof. Let $(w_i)_{i\geq 1}$ be an ordered basis of W. Let's construct an ordered basis $(v_i)_{i\geq 1}$ of V recursively in the following way. For all $k \geq 1$ let $f(k)$ be the least integer such that $w_{f(k)} \notin \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<k})$, and let v'_k be an element of least norm in $V \setminus \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<k})$. If $\|v'_k\| < \|w_{f(k)}\|$, we set $v_k \coloneqq v'_k$, otherwise we set $v_k \coloneqq w_{f(k)}$.

By construction $(v_k)_{k\geq 1}$ is an ordered basis of V, so we only need to show that for all $j \geq 1$ there is some $k \geq 1$ such that $v_k = w_j$. By contradiction, let j be the first integer such that this does not happen, and let k be the greatest integer such that $w_j \notin \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<k})$, which exists because $(v_i)_{i\geq 1}$ is a basis of V. This means that $w_j = \alpha v_k + v$ for some $v \in \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i<k})$ and some constant $\alpha \in \mathbb{F}_q^\times$, and since $v_k \neq w_j$, by our algorithm we must have $\|v_k\| < \|w_j\|$; as a consequence $\|v\| = \|w_j - \alpha v_k\| = \|w_j\| > \|v_k\|$, which is a contradiction because, since $(v_i)_{i\geq 1}$ is an ordered basis, $\|v_k\| \geq \|v_i\|$ for all $i < k$, hence $\|v_k\| \geq \|v\|$.

If $\dim_{\mathbb{F}_q} {}^{V}\!/_{W} = n < \infty$, since the basis $\{v_i\}_{i\geq 1}$ of V extends the basis $\{w_i\}_{i\geq 1}$ of W, there are exactly n elements of the former which are not contained in the latter. Since, taking the order into account, $(w_i)_{i\geq 1}$ is a subsequence of $(v_i)_{i\geq 1}$, for $i \gg 0$ we have $v_i = w_{i+n}$, hence $r_i = s_{i+n}$. □

We now consider $\mathbf{C} = \mathbb{C}_\infty$ with the usual norm $\|\cdot\|$.

Lemma 5.1.12. *Let $V \subseteq \mathbb{C}_\infty$ be a lattice which is also a (projective) A-submodule of finite rank r, and let $(s_i)_{i\geq 1}$ be its norm sequence. Then:*

- *for all $i \gg 0$, $s_{i+er} = q^e \cdot s_i$;*
- *for all $k \in \mathbb{Z}$, for all $i \gg 0$, $\frac{s_{i+k}}{s_i} \leq q^{e\lceil \frac{k}{er} \rceil}$;*
- *for all $k \in \mathbb{Z}$, for infinitely many i, $\frac{s_{i+k}}{s_i} \leq q^{\frac{k}{r}}$.*

Proof. We can choose $a, b \in A \setminus \{0\}$ such that $\deg(b) = \deg(a) + e$. Fix an ordered basis $(v_i)_{i\geq 1}$ of V: obviously, $(av_i)_{i\geq 1}$ and $(bv_i)_{i\geq 1}$ are ordered bases respectively of aV and bV. Since V has rank r, $\dim_{\mathbb{F}_q} {}^{V}\!/_{aV} = r\deg(a)$ and $\dim_{\mathbb{F}_q} {}^{V}\!/_{bV} = r\deg(b)$, therefore by Lemma 5.1.11 we have $\|v_i\| = \|av_{i-r\deg(a)}\| = \|bv_{i-r\deg(b)}\|$ for $i \gg 0$. Rearranging the terms, we get that, for $i \gg 0$:

$$\|v_{i-r\deg(a)}\| = \|a\|^{-1}\|b\|\|v_{i-r\deg(b)}\| = q^e\|v_{i-r\deg(b)}\|.$$

Shifting the pedices we get $\|v_i\| = q^e\|v_{i-r(\deg(b)-\deg(a))}\| = q^e\|v_{i-er}\|$ for $i \gg 0$, which is the first statement.

For all $k \in \mathbb{Z}$, since the norm sequence is weakly increasing, we have the following inequality for $i \gg 0$:

$$\frac{s_{i+k}}{s_i} \leq \frac{s_{i+er\lceil \frac{k}{er} \rceil}}{s_i} = q^{e\lceil \frac{k}{er} \rceil}.$$

Moreover, for all $i \gg 0$:

$$\prod_{j=0}^{er-1} \frac{s_{i+k(j+1)}}{s_{i+kj}} = \frac{s_{i+ker}}{s_i} = \begin{cases} \prod_{j=0}^{k-1} \frac{s_{i+e(j+1)r}}{s_{i+ejr}} = q^{ek} \text{ if } k \geq 0 \\ \prod_{j=k}^{-1} \frac{s_{i+e(j+1)r}}{s_{i+ejr}} = q^{ek} \text{ if } k < 0 \end{cases},$$

hence at least one of the factors on the left hand side has norm at most $q^{\frac{k}{r}}$; this implies that the inequality $\frac{s_{i+k}}{s_i} \leq q^{\frac{k}{r}}$ holds for infinitely many values of i.

□

5.1.3 Estimation of the coefficients of g_β and $\exp_\phi$

Definition 5.1.13. For a lattice $V \subseteq \mathbb{C}_\infty$, for all integers $i \geq 0$ we define:

$$e_{V,i} := \sum_{\substack{I \subseteq V \setminus 0 \\ |I| = q^i - 1}} \prod_{v \in I} v^{-1}$$

(by convention, $e_{V,0} = 1$).

Remark 5.1.14. For all $c \in \mathbb{C}_\infty$, since $V \subseteq \mathbb{C}_\infty$ is a lattice, the infinite product

$$c \prod_{v \in V} \left(1 - \frac{c}{v}\right)$$

converges, and is equal to

$$\sum_{n \geq 0} e_{V,n} c^{q^n}.$$

In particular,

$$\sum_{n \geq 0} e_{V,n} x^{q^n} \in \mathbb{C}_\infty[[x]]$$

is the only power series with infinite radius of convergence and with leading coefficient 1 such that its zeroes are simple and coincide with V.

Lemma 5.1.15. *Fix a lattice $V \subseteq \mathbb{C}_\infty$, with norm sequence $(r_i)_{i \geq 1}$. Fix an ordered basis $(v_i)_{i \geq 1}$ and call $V_m := \mathrm{Span}_{\mathbb{F}_q}(\{v_i\}_{i \leq m})$ for all $m \geq 0$. We have:*

- *for all $k \geq 0$:*

$$\|e_{V,k}\| \leq \prod_{i=1}^{k} r_i^{q^{i-1} - q^i};$$

- *for all $m > 0$, for all $k > 0$:*

$$\left\| \sum_{v \in V_m} v^{q^k - 1} \right\| \begin{cases} = 0 & \text{if } k < m \\ \leq r_m^{q^k - q^m} \prod_{i=1}^m r_i^{q^i - q^{i-1}} & \text{if } k \geq m \end{cases}.$$

Proof. For the first part, if $k = 0$ then $e_{V,k} = 1$, so there is nothing to prove. If $k > 0$, we have:

$$\|e_{V,k}\| = \left\| \sum_{\substack{I \subseteq V \setminus \{0\} \\ |I| = q^k - 1}} \prod_{v \in I} v^{-1} \right\| \leq \max_{\substack{I \subseteq V \setminus \{0\} \\ |I| = q^k - 1}} \left\| \prod_{v \in I} v^{-1} \right\| = \left\| \prod_{v \in V_k} v^{-1} \right\| = \prod_{i=1}^{k} r_i^{q^{i-1} - q^i}.$$

For the second part note that, in the notation of Lemma 4.1.6, the element whose norm we are considering is equal to $S_{m,q^k-1}(v_1,\dots,v_m)$. By that lemma, if $k<m$, the element is zero, otherwise we have the following inequality:

$$\|S_{m,q^k-1}(v_1,\dots,v_m)\| \leq \max_{\substack{d_1,\dots,d_m \\ d_1+\cdots+d_m=q^k-1 \\ \forall j\ d_1+\cdots+d_j\geq q^j-1}} \left\|v_1^{d_1}\cdots v_m^{d_m}\right\|.$$

It's easy to see that the maximum norm of the product $v_1^{d_1}\cdots v_m^{d_m}$ under the specified conditions is obtained when we set $d_i=q^i-q^{i-1}$ for $i<m$ and $d_m=q^k-q^{m-1}$, therefore we get the desired inequality. □

Let ϕ be a Drinfeld module of rank r with lattice Λ_ϕ.

Remark 5.1.16. Since $\Lambda_\phi\subseteq K_\infty\Lambda_\phi$ is discrete and $K_\infty\Lambda_\phi\cong K_\infty^r$ is locally compact, Λ_ϕ is a lattice of $\mathbb{C}_\infty$. Moreover, $e_{\Lambda_\phi,n}$ is exactly the coefficient of τ^n of the exponential function $\exp_\phi\in\mathbb{C}_\infty[[\tau]]$.

Recall the definition of g_β for all $\beta\in\ker(\exp_\phi^*)$ given in Lemma 5.1.4.

Lemma 5.1.17. *For all $\beta\in\ker(\exp_\phi^*)\setminus\{0\}$, $\ker(g_\beta)$ is an $\mathbb{F}_q$-vector subspace of Λ_ϕ of codimension 1. In particular, $g_\beta=\beta\sum_{n\geq0}e_{\ker(g_\beta),n}\tau^n$.*

Proof. If $c\in\ker(g_\beta)$ then $\exp_\phi(c)=\beta^{-1}(1-\tau)(g_\beta(c))=0$, hence $c\in\Lambda_\phi$. Moreover, $g_\beta|_{\Lambda_\phi}$ is an $\mathbb{F}_q$-linear function with image in $\mathbb{F}_q$, hence its kernel V_β has codimension at most 1 in Λ_ϕ. It is exactly 1 because $g_\beta|_{\Lambda_\phi}$ is not identically zero by Theorem 5.1.6.

From the identity $(1-\tau)\circ g_\beta=\beta\exp_\phi$, since the zeroes of $\exp_\phi$ are simple, we deduce the same for the zeroes of g_β, therefore $g_\beta=c_\beta\sum_{n\geq0}e_{\ker(g_\beta),n}\tau^n$ for some constant $c_\beta\in\mathbb{C}_\infty$ by Remark 5.1.14. Finally, from the same identity we deduce that the coefficient of τ in the expansion of g_β is β, hence $c_\beta=\beta$. □

5.1.4 Proof of the identities

We can now prove the main propositions of this section.

Proposition 5.1.18. *For all $\beta\in\ker(\exp_\phi^*)$, the following identity holds in $\mathbb{C}_\infty$:*

$$\beta=-\sum_{\lambda\in\Lambda_\phi\setminus\{0\}}\frac{g_\beta(\lambda)}{\lambda}.$$

Proof. The series converges for all $\beta\in\ker(\exp_\phi^*)$ because the denominators belong to the lattice Λ_ϕ and the numerators to $\mathbb{F}_q$. For $\beta=0$ the identity is obvious, hence we can suppose $\beta\neq0$. Fix an ordered basis $(\lambda_i)_{i\geq1}$ of Λ_ϕ and define $\Lambda_m:=\mathrm{Span}_{\mathbb{F}_q}(\{\lambda_i\}_{i\leq m})$ for all $m\geq0$. By Lemma 5.1.17, $\ker(g_\beta)\subseteq\Lambda_\phi$ has codimension 1, hence by Lemma 5.1.11, if we denote by $(r_i)_{i\geq1}$ and $(s_i)_{i\geq1}$ the norm sequences respectively of Λ_ϕ and $\ker(g_\beta)$, there is a positive integer N such that for all $i<N$ $s_i=r_i$, and for all $i\geq N$ $s_i=r_{i+1}$. For all $m\geq N$, we define:

$$S_m:=\beta+\sum_{\lambda\in\Lambda_m\setminus\{0\}}\frac{g_\beta(\lambda)}{\lambda}=\beta\sum_{k\geq1}e_{\ker(g_\beta),k}\sum_{\lambda\in\Lambda_m}\lambda^{q^k-1}.$$

By Lemma 5.1.15, we have:

$$\begin{aligned}
\|\beta^{-1}S_m\| &= \left\|\sum_{k\geq 1} e_{\ker(g_\beta),k} \sum_{\lambda\in\Lambda_m} \lambda^{q^k-1}\right\| \leq \max_{k\geq m}\left\{\|e_{\ker(g_\beta),k}\| \left\|\sum_{\lambda\in\Lambda_m} \lambda^{q^k-1}\right\|\right\} \\
&\leq \max_{k\geq m}\left\{\left(\prod_{i=1}^{k} s_i^{q^{i-1}-q^i}\right)\left(r_m^{q^k-q^m}\prod_{i=1}^{m} r_i^{q^i-q^{i-1}}\right)\right\} \\
&= \max_{k\geq m}\left\{\left(\prod_{i=N}^{k} r_{i+1}^{q^{i-1}-q^i}\right)\left(r_m^{q^k-q^m}\prod_{i=N}^{m} r_i^{q^i-q^{i-1}}\right)\right\} \\
&= \max_{k\geq m}\left\{\left(\prod_{i=N}^{m}\left(\frac{r_i}{r_{i+1}}\right)^{q^i-q^{i-1}}\right)\left(\prod_{i=m+1}^{k}\left(\frac{r_m}{r_i}\right)^{q^i-q^{i-1}}\right)\right\} \\
&= \prod_{i=N}^{m}\left(\frac{r_i}{r_{i+1}}\right)^{q^i-q^{i-1}} \\
&= \left(\frac{r_N}{r_{m+1}}\right)^{q^N-q^{N-1}} \prod_{i=N+1}^{m}\left(\frac{r_i}{r_{m+1}}\right)^{q^i-2q^{i-1}+q^{i-2}} \\
&\leq \left(\frac{r_N}{r_{m+1}}\right)^{q^N-q^{N-1}}.
\end{aligned}$$

Since this number tends to zero as m tends to infinity, we have the following identity in $\mathbb{C}_\infty$:

$$0 = \lim_m S_m = \lim_m \left(\beta + \sum_{\lambda\in\Lambda_m\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda}\right) = \beta + \sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{g_\beta(\lambda)}{\lambda}. \qquad \square$$

Proposition 5.1.19. *For all integers k, for all $c \in K_\infty \setminus \{0\}$ with $\|c\| \leq q^{\frac{k-1}{r}}$, the following identity holds in $\mathbb{C}_\infty$:*

$$\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\exp_\phi(c\lambda)}{\lambda^{q^k}} = -\sum_{i+j=k} e_j l_i^{q^j} c^{q^j}.$$

Proof. First of all, let's show that series on the left hand side converges. Since $\exp_\phi(K_\infty\Lambda_\phi)$ is homeomorphic to the compact space $K_\infty\Lambda_\phi/\Lambda_\phi$, the numerators $\exp_\phi(c\lambda)$ are bounded in norm by some positive real constant. In particular, since $\Lambda_\phi \subseteq \mathbb{C}_\infty$ is a lattice, for any positive real number ε there are finitely many $\lambda \in \Lambda_\phi$ such that $\left\|\frac{\exp_\phi(c\lambda)}{\lambda^{q^k}}\right\| < \varepsilon$, so the series converges.

Fix an ordered basis $(\lambda_i)_{i\geq 1}$ of Λ_ϕ, set $r_i := \|\lambda_i\|$, and define $\Lambda_m := \mathrm{Span}_{\mathbb{F}_q}(\{\lambda_i\}_{i\leq m})$ for all $m \geq 1$; define:

$$S_m := \sum_{\lambda\in\Lambda_m\setminus\{0\}} \frac{\exp_\phi(c\lambda)}{\lambda^{q^k}} - \sum_{\substack{0\leq j\leq k \\ \lambda\in\Lambda_m\setminus\{0\}}} e_j c^{q^j}\lambda^{q^j-q^k} = \sum_{j\geq 1} e_{k+j} c^{q^{k+j}} \left(\sum_{\lambda\in\Lambda_m} \lambda^{q^j-1}\right)^{q^k},$$

where by convention we set $e_j = 0$ for all $j < 0$. By Lemma 5.1.15, for all $m \gg 0$ we have:

$$\begin{aligned}
\|S_m\| =& \left\| \sum_{j\geq 1} e_{k+j} c^{q^{k+j}} \left(\sum_{\lambda\in\Lambda_m} \lambda^{q^j-1} \right)^{q^k} \right\| \leq \max_{j\geq m} \left\{ \|e_{k+j}\| \|c\|^{q^{k+j}} \left\| \sum_{\lambda\in\Lambda_m} \lambda^{q^j-1} \right\|^{q^k} \right\} \\
\leq& \max_{j\geq m} \left\{ \|c\|^{q^{k+j}} \left(\prod_{i=1}^{k+j} r_i^{q^{i-1}-q^i} \right) \left(r_m^{q^j-q^m} \prod_{i=1}^{m} r_i^{q^i-q^{i-1}} \right)^{q^k} \right\} \\
\leq& \max_{j\geq m} \left\{ \|c\|^{q^k} \left(\prod_{i=1-k}^{j} r_{i+k}^{q^{i+k-1}-q^{i+k}} \right) \left(\prod_{i=1}^{j} (\|c\| r_i)^{q^{i+k}-q^{i+k-1}} \right) \right\} \\
=& C_k \cdot \max_{j\geq m} \left\{ \prod_{i=m+1}^{j} \left(\frac{\|c\| r_i}{r_{i+k}} \right)^{q^{i+k}-q^{i+k-1}} \right\} \\
&\implies \limsup_m \|S_m\| \leq C_k \cdot \limsup_j \prod_{i=m+1}^{j} \left(\frac{\|c\| r_i}{r_{i+k}} \right)^{q^{i+k}-q^{i+k-1}},
\end{aligned}$$

where C_k is a nonzero constant which depends on k. Since the norms of nonzero elements of K_∞ are integer powers of q^e, we actually have the inequality $\|c\| \leq q^{e\left\lfloor \frac{k-1}{er} \right\rfloor}$. By Lemma 5.1.12, we have:

$$\begin{aligned}
\|c\| \frac{r_i}{r_{i+k}} &\leq q^{e\left\lfloor \frac{k-1}{er} \right\rfloor} \cdot q^{e\left\lceil -\frac{k}{er} \right\rceil} = q^{e\left\lfloor \frac{k-1}{er} \right\rfloor} \cdot q^{-e\left\lfloor \frac{k}{er} \right\rfloor} \leq 1 \text{ for all } i \text{ large enough;} \\
\|c\| \frac{r_i}{r_{i+k}} &\leq q^{e\left\lfloor \frac{k-1}{er} \right\rfloor} \cdot q^{-\frac{k}{r}} \leq q^{\frac{k-1}{r}} \cdot q^{-\frac{k}{r}} = q^{-\frac{1}{r}} < 1 \text{ for infinitely many values of } i.
\end{aligned}$$

The first inequality implies that the limit superior

$$\limsup_j \prod_{i=m+1}^{j} \left(\frac{\|c\| r_i}{r_{i+k}} \right)^{q^{i+k}-q^{i+k-1}}$$

is finite, the second inequality implies that it is zero. We deduce that the sequence $\|S_m\|$ converges to 0. If $k < 0$, we get the following identity in $\mathbb{C}_\infty$:

$$\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\exp_\phi(c\lambda)}{\lambda^{q^k}} = \lim_m S_m = 0.$$

If instead $k \geq 0$, we get the following identity in $\mathbb{C}_\infty$:

$$\begin{aligned}
\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\exp_\phi(c\lambda)}{\lambda^{q^k}} &= \lim_m \left(S_m + \sum_{j=0}^{k} e_j c^{q^j} \sum_{\lambda\in\Lambda_m\setminus\{0\}} \lambda^{q^j-q^k} \right) \\
&= \sum_{j=0}^{k-1} \left(e_j c^{q^j} \sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \lambda^{q^j-q^k} \right) - e_k c^{q^k} \\
&= -\sum_{j=0}^{k} e_j l_{k-j}^{q^j} c^{q^j}
\end{aligned}$$

where the last equality follows from Lemma 4.3.21. □

5.2 Universal dual Anderson eigenvector

Recall the definition and properties of Anderson modules discussed in Subsection 2.2.1. Let's formulate a notion of *dual Anderson eigenvectors*, analogously to Definition 2.2.7, for any Anderson A-module $\underline{E} = (E, \phi)$.

Recall that by Remark 2.2.4 $E(\mathbb{C}_\infty)$ has a natural structure of topological $\mathbb{C}_\infty$-module, hence the same holds for $E(\mathbb{C}_\infty)^\vee := \mathrm{Hom}_{\mathbb{C}_\infty}(E(\mathbb{C}_\infty), \mathbb{C}_\infty)$. Let's endow the $E(\mathbb{C}_\infty)$ with the natural A-module structure induced by ϕ.

Definition 5.2.1. For all integers k, for any pair of $\mathbb{C}_\infty$-vector spaces V, W let's set:

$$\mathrm{Hom}_k(V,W) := \{f \in \mathrm{Hom}_{\mathbb{F}_q}(V,W) | \forall c \in \mathbb{C}_\infty \ f \circ c = c^{q^k} \circ f\},$$

$$\underline{\mathrm{Hom}}(V,W) := \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_k(V,W);$$

similarly, $\mathrm{End}_k(V) := \mathrm{Hom}_k(V,V)$ and $\underline{\mathrm{End}}(V) := \underline{\mathrm{Hom}}(V,V)$. Moreover, we define the adjoint $\cdot^* : \mathrm{Hom}_k(V,W) \to \mathrm{Hom}_{-k}(W^\vee, V^\vee)$ sending f to the map f^* taking $h \in W^\vee$ to $\tau^{-k} \circ h \circ f$.

Lemma 5.2.2. *Let $V := \mathbb{C}_\infty^n$ and $W := \mathbb{C}_\infty^m$. Each function in $\mathrm{Hom}_k(V,W)$ can be written uniquely as $M\tau^k$, where M is an m-by-n matrix with coefficients in $\mathbb{C}_\infty$ and $\tau : V \to V$ sends the vector $(v_i)_i$ to $(v_i^q)_i$.*

Proof. Fix $f \in \mathrm{Hom}_k(V,W)$. For all $v \in V$ and for all $c \in \mathbb{C}_\infty$ we have:

$$f \circ \tau^{-k}(cv) = f\left(c^{q^{-k}} \tau^{-k}(v)\right) = c\left(f \circ \tau^{-k}(v)\right),$$

hence $f \circ \tau^{-k}$ can be represented by an m-by-n matrix. □

Corollary 5.2.3. *If V and W are finite topological $\mathbb{C}_\infty$-vector spaces, all functions in $\mathrm{Hom}_k(V,W)$ are continuous. In particular, there is a natural immersion from $\underline{\mathrm{Hom}}(V,W)$ to the set of continuous $\mathbb{F}_q$-linear homomorphisms from V to W.*

Remark 5.2.4. We can extend the adjoint to a map $* : \underline{\mathrm{Hom}}(V,W) \to \underline{\mathrm{Hom}}(W^\vee, V^\vee)$. If V, W are finite $\mathbb{C}_\infty$-vector spaces, the adjoint is a bijection. If moreover $V = W$, the space $\underline{\mathrm{End}}(V) := \underline{\mathrm{Hom}}(V,V)$ has a natural noncommutative algebra structure induced by the composition, and $* : \underline{\mathrm{End}}(V) \to \underline{\mathrm{End}}(V^\vee)$ is an anti-isomorphism of algebras.

Remark 5.2.5. The algebra $\bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathrm{End}_k(E(\mathbb{C}_\infty))$ is naturally isomorphic to the algebra of endomorphisms of E as an $\mathbb{F}_q$-module scheme over $\mathbb{C}_\infty$ (see for example [Gos98][Lemma 5.4.4]).

Definition 5.2.6. The action $\phi^* : A \to \underline{\mathrm{End}}(E(\mathbb{C}_\infty)^\vee) \subseteq \mathrm{End}_{\mathbb{F}_q}^{cont}(E(\mathbb{C}_\infty)^\vee)$ is defined as the adjoint of $\phi : A \to \underline{\mathrm{End}}(E(\mathbb{C}_\infty))$.

Remark 5.2.7. For all $a \in A$, if we write $\phi_a = \sum_k (\phi_a)_k \in \underline{\mathrm{End}}(E(\mathbb{C}_\infty))$, we have that $\phi_a^* = \sum_k (\phi_a)_k^*$.

In the special case of Drinfeld modules, where we take $E = \mathbb{G}_a$, this notation agrees with the one established in Proposition 4.3.11.

Definition 5.2.8. Let $\underline{E} = (E, \phi)$ be an Anderson A-module. For any discrete A-module M, its set of *dual Anderson eigenvectors* is defined as the A-module of continuous A-linear homomorphisms $\mathrm{Hom}_A^{cont}(\hat{M}, E(\mathbb{C}_\infty)^\vee) \subseteq E(\mathbb{C}_\infty)^\vee \hat{\otimes} M$. We denote by $\mathrm{Sf}_{\phi^*} : A - \mathbf{Mod} \to A - \mathbf{Mod}$ the natural functor that extends this map.

Remark 5.2.9. We can write as follows the property of being a dual Anderson eigenvector $\zeta = \sum_i z_i \otimes m_i \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} M$. For all $a \in A$:

$$\sum_i z_i \otimes am_i = \left(\sum_j (\phi_a)_j^* \otimes 1\right)\left(\sum_i z_i \otimes m_i\right) = \sum_{i,j} (\phi_a)_j^* z_i \otimes m_i = \sum_{i,j} (\tau^{-j} \circ z_i \circ (\phi_a)_j) \otimes m_i.$$

From now on, let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld module. In this case, we can naturally identify $\mathbb{G}_a(\mathbb{C}_\infty)^\vee$ with $\mathbb{C}_\infty$. To avoid confusion, we denote this object by $\mathbb{C}_\infty^{\phi^*}$, to underline its A-module structure. If for any $a \in A$ we write $\phi_a = \sum_i a_i \tau^i$, under this action a sends $c \in \mathbb{C}_\infty^{\phi^*}$ to $\phi_a^*(c) = \sum_i a_i^{q^{-i}} c^{q^{-i}}$.

Theorem 5.2.10. *Let $(\mathbb{G}_a, \phi)$ be a Drinfeld module of rank r. The functor Sf_{ϕ^*} is naturally isomorphic to $\mathrm{Hom}_A(\Lambda_\phi, \_)$; moreover, the universal object in $\mathbb{C}_\infty^{\phi^*} \hat{\otimes} \Lambda_\phi$ corresponds to the map $\hat{\Lambda}_\phi \cong \ker \exp_\phi^* \subseteq \mathbb{C}_\infty^{\phi^*}$ and can be expressed as $-\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda$.*

Proof. The map $\exp_\phi^* : \mathbb{C}_\infty^{\phi^*} \to \mathbb{C}_\infty$ is a continuous A-linear morphism; for any discrete A-module M, it induces a morphism $\mathrm{Sf}_{\phi^*}(M) \to \mathrm{Hom}_A^{cont}(\hat{M}, \mathbb{C}_\infty)$, where the Pontryagin dual $\hat{M} = \mathrm{Hom}_{\mathbb{F}_q}(M, \mathbb{F}_q)$ is endowed with its natural structure of compact A-module. Fix some $\zeta \in \mathrm{Sf}_{\phi^*}(M)$, with image $\overline{\zeta}$: since $\hat{M}$ is compact, the image of $\overline{\zeta}$ must be a compact sub-A-module of $\mathbb{C}_\infty$, but for any $c \in \mathbb{C}_\infty \setminus \{0\}$ the set $A \cdot c$ is unbounded, hence $\overline{\zeta} \equiv 0$. We deduce that the image of $\zeta : \hat{M} \to \mathbb{C}_\infty^{\phi^*}$ must be contained in $\ker \exp_\phi^*$, which by Theorem 5.1.6 is isomorphic as a topological A-module to $\hat{\Lambda}_\phi$; we have the following natural isomorphisms:

$$\mathrm{Sf}_{\phi^*}(M) = \mathrm{Hom}_A^{cont}(\hat{M}, \ker \exp_\phi^*) \cong \mathrm{Hom}_A^{cont}\left(\widehat{\ker \exp_\phi^*}, M\right) \cong \mathrm{Hom}_A(\Lambda_\phi, M),$$

where we used Lemma 2.1.6 for the second isomorphism.

The universal object $\zeta_\phi \in \mathbb{C}_\infty^{\phi^*} \hat{\otimes} \Lambda_\phi$ is given by the natural morphism

$$\psi : \hat{\Lambda}_\phi \cong \ker \exp_\phi^* \subseteq \mathbb{C}_\infty^{\phi^*}$$

of Theorem 5.1.6, which by Proposition 5.1.18 sends $g \in \hat{\Lambda}_\phi$ to $-\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{g(\lambda)}{\lambda}$.

If we fix an $\mathbb{F}_q$-basis $(\lambda_i)_i$ of Λ_ϕ, with $(\lambda_i^*)_i$ dual basis of $\hat{\Lambda}_\phi$, by Proposition 2.1.14 we can write $\zeta_\phi = \sum_i \psi(\lambda_i^*) \otimes \lambda_i$, hence:

$$\zeta_\phi = \sum_i \left(- \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{\lambda_i^*(\lambda)}{\lambda} \right) \otimes \lambda_i = - \sum_{\lambda \in \Lambda_\phi \setminus \{0\}, i} \lambda^{-1} \otimes \lambda_i^*(\lambda) \lambda_i = - \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes \lambda.$$

□

Definition 5.2.11. We define the *universal dual Anderson eigenvector* $\zeta_\phi \in \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ as the universal object of the functor Sf_{ϕ^*}.

Corollary 5.2.12. *For all discrete A-modules M, $\mathrm{Sf}_{\phi^*}(M)$ is isomorphic to $\mathrm{Hom}_A(\Lambda_\phi, M)$ as an $A \otimes A$-module. In particular, for any M we have the following equality between subsets of $\mathbb{C}_\infty \hat{\otimes} M$:*

$$\mathrm{Sf}_{\phi^*}(M) = \left\{ \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \lambda^{-1} \otimes l(\lambda) \middle| l \in \mathrm{Hom}_A(\Lambda_\phi, M) \right\}.$$

Remark 5.2.13. Fix an $\mathbb{F}_q$-basis $(\lambda_i)_i$ of the discrete A-module Λ_ϕ, with $(\lambda_i^*)_i$ dual basis of $\hat{\Lambda}_\phi$. By Proposition 2.1.14 we can express the universal object in the following alternative way as an element of $\mathbb{C}_\infty^{\phi^*}\hat{\otimes}\Lambda_\phi$:

$$\zeta_\phi = \sum_i \psi(\lambda_i^*) \otimes \lambda_i,$$

where ψ denotes Poonen's isomorphism $\hat{\Lambda}_\phi \cong \ker(\exp_\phi^*) \subseteq \mathbb{C}_\infty^{\phi^*}$.

5.3 A convergence result for the universal Anderson eigenvector

Let's fix a Drinfeld module $(\mathbb{G}_a, \phi)$ of rank 1 with period lattice Λ_ϕ, and write $\zeta_\phi = \sum_i z_i \otimes \lambda_i \in \mathbb{C}_\infty\hat{\otimes}\Lambda_\phi$, where $(\lambda_i)_{i\geq 1}$ is an ordered basis of Λ_ϕ.If we assume $\infty \in X(\mathbb{F}_q)$, Chung, Ngo Dac, and Pellarin proved that, for any nonnegative integer k, $\sum_i z_i^{q^k}\lambda_i$ converges to the k-th coefficient of the logarithm, while for any negative integer k it converges to 0 ([CNP23]).

We aim to prove this result for a Drinfeld module of arbitrary rank, exploiting the defining property of the universal Anderson eigenvector.

Proposition 5.3.1. *Let $(\mathbb{G}_a, \phi)$ be a Drinfeld module of rank r, fix an $\mathbb{F}_q$-linear basis $(\lambda_i)_{i\geq 1}$ of Λ_ϕ, and write $\zeta_\phi = \sum_i z_i \otimes \lambda_i \in \mathbb{C}_\infty\hat{\otimes}\Lambda_\phi$. Then, for all integers k the series $\sum_i z_i^{q^k}\lambda_i$ converges; moreover, if $k \geq 0$ it converges to the k-th coefficient of the logarithm l_k, while if $k < 0$ it converges to* 0.

Proof. Let's fix $a \in A \setminus \mathbb{F}_q$ (since X is geometrically irreducible, $\deg(a) > 0$) and fix an ordered basis $(\lambda_i'')_{i\geq 1}$ of Λ_ϕ. By Lemma 5.1.12 there is some N such that, for all $i > N$, $\|\lambda''_{i+r\deg(a)}\| = \|a\lambda_i''\|$; for $i > 0$, let's define

$$\lambda'_{i+N} := \begin{cases} \lambda''_{i+N} \text{ if } 1 \leq i \leq r\deg(a) \\ a\lambda'_{i+N-r\deg(a)} \text{ if } i > r\deg(a) \end{cases},$$

so that $\|\lambda_i'\| = \|\lambda_i''\|$ for all $i > N$. By Lemma 5.1.12, the rank of $\Lambda' := \mathrm{Span}_{\mathbb{F}_q}(\{\lambda_i'\}_{i>N}) \subseteq \Lambda_\phi$ as an $\mathbb{F}_q[a]$-module is $r\deg(a)$, hence the elements $\{\lambda_i'\}_{N<i\leq N+r\deg(a)}$, which generate Λ' as an $\mathbb{F}_q[a]$-module, are $\mathbb{F}_q[a]$-linearly independent. In particular, the sequence $(\lambda_i')_{i>N}$ forms an ordered basis of Λ', hence, by Lemma 5.1.11, Λ' has codimension N in Λ_ϕ. We choose $\lambda_1', \dots, \lambda_N' \in \Lambda_\phi$ to extend the basis of Λ' to Λ_ϕ.

Let's write $\zeta_\phi = \sum_i z_i' \otimes \lambda_i'$; if we denote by $(\lambda_i'^*)_i$ the corresponding dual basis of $\hat{\Lambda}_\phi$, and if we call ψ Poonen's isomorphism $\hat{\Lambda}_\phi \cong \ker(\exp_\phi^*) \subseteq \mathbb{C}_\infty$, by Remark 5.2.13 we know that $z_i' = \psi(\lambda_i'^*)$; in particular, for $i \gg 0$, we have

$$z_i' = \psi(\lambda_i'^*) = \psi(a\lambda'_{i+r\deg(a)}{}^*) = \phi_a^*\left(\psi(\lambda'_{i+r\deg(a)}{}^*)\right) = \phi_a^*(z'_{i+r\deg(a)}).$$

Let's write $\phi_a^* = \sum_k \tau^{-k}a_k$. There is some real constant $\varepsilon > 0$ such that that, for any $c \in \mathbb{C}_\infty$ with $\|c\| < \varepsilon$, $\|\phi_a^*(c)\| = \|a_{r\deg(a)}c\|^{q^{-r\deg(a)}}$. Since the sequence $(z_i')_i$ converges to 0, for $i \gg 0$ we have $\|z_i'\| = \|\phi_a^*(z'_{i+r\deg(a)})\| = \|a_{r\deg(a)}z'_{i+r\deg(a)}\|^{q^{-r\deg(a)}}$, hence $\|z'_{i+r\deg(a)}\| = \|z_i'\|^{q^{r\deg(a)}}\|a_{r\deg(a)}\|^{-1}$. For any positive real constant $\varepsilon < 1$ there is a positive integer M such that, for all $i > M$, $\|z_i'\|\|a_{r\deg(a)}\|^{-\frac{1}{q-1}} < \varepsilon$. By recursion—assuming M is large enough—we deduce that, for all $k \geq 0$ and for all $i > M$:

$$\|z'_{i+kr\deg(a)}\| = \|z_i'\|^{q^{kr\deg(a)}}\|a_{r\deg(a)}\|^{-\frac{q^{kr\deg(a)}-1}{q-1}} < \varepsilon^{q^{kr\deg(a)}}\|a_{r\deg(a)}\|^{\frac{1}{q-1}}.$$

In particular, by setting $i = M+1, \dots, M + r\deg(a)$, and setting $\delta := \varepsilon^{q^{-M-r\deg(a)}} < 1$, we deduce that $\|z'_n\| < \delta^{q^n} \|a_{r\deg(a)}\|^{\frac{1}{q-1}}$ for $n \geq M$. Since $\lambda'_{n+r\deg(a)} = a\lambda'_n$ for $n \gg 0$, there is a real constant $C > 0$ such that $\|\lambda'_n\| < C\|a\|^{\frac{n}{r\deg(a)}} = Cq^{\frac{n}{r}}$ for all n, hence for all $k \in \mathbb{Z}$ we have

$$\limsup_n \|z'_n\|^{q^k} \|\lambda'_n\| < C\|a_{r\deg(a)}\|^{\frac{1}{q-1}} \cdot \limsup_n \delta^{q^{n+k}} q^{\frac{n}{r}} = 0.$$

We deduce that the series $\sum_n z'^{q^k}_n \lambda'_n$ converges in $\mathbb{C}_\infty$ for any integer k.

For all i, we can write λ'_i as a finite sum $\sum_j \alpha_{i,j}\lambda_j$ with constants $\alpha_{i,j} \in \mathbb{F}_q$, so we have:

$$\zeta_\phi = \sum_i z'_i \otimes \lambda'_i = \sum_i z'_i \otimes \left(\sum_j \alpha_{i,j}\lambda_j\right) = \sum_i \sum_j \alpha_{i,j} z'_i \otimes \lambda_j = \sum_j \left(\sum_i \alpha_{i,j} z'_i\right) \otimes \lambda_j.$$

For all j, we deduce $z_j = \sum_i \alpha_{i,j} z'_i$. Moreover, for any integer k:

$$\sum_j z_j^{q^k} \lambda_j = \sum_j \left(\sum_{i\geq j} \alpha_{i,j} z'^{q^k}_i\right) \lambda_j = \sum_i z'^{q^k}_i \left(\sum_{j\leq i} \alpha_{i,j}\lambda_j\right) = \sum_i z'^{q^k}_i \lambda'_i.$$

For all k, let's set $l'_k := \sum_i z'^{q^k}_i \lambda'_i$. If $k > 0$, we have:

$$\begin{aligned} l_k = \sum_{\lambda\in\Lambda_\phi} \lambda^{1-q^k} &= \sum_{\lambda\in\Lambda_\phi} \lambda^{-q^k} \sum_i \lambda'^*_i(\lambda)\lambda'_i = \sum_i \left(\sum_{\lambda\in\Lambda_\phi} \lambda^{-q^k} \lambda'^*_i(\lambda)\right) \lambda'_i \\ &= \sum_i \left(\sum_{\lambda\in\Lambda_\phi} \lambda^{-1}\lambda'^*_i(\lambda)\right)^{q^k} \lambda'_i = \sum_i g(\lambda'^*_i)^{q^k} \lambda'_i = \sum_i z'^{q^k}_i \lambda'_i = l'_k. \end{aligned}$$

Note that for all $a \in A$, $\sum_i z'_i \otimes a\lambda_i = \sum_i \phi^*_a(z'_i) \otimes \lambda_i$, hence for any integer k we have the identity:

$$\sum_i z'^{q^k}_i \lambda_i = \sum_i \phi^*_a(z'_i)^{q^k} \lambda_i.$$

Define $\log'_\phi := \sum_k l'_k \tau^k \in \mathbb{C}_\infty[[\tau^{-1}, \tau]]$. For all $a \in A$, if we write $\phi^*_a = \sum_j \tau^{-j} a_j$, we have:

$$\begin{aligned} a\log'_\phi = a\sum_k l'_k \tau^k &= \sum_k \sum_i a\lambda'_i z'^{q^k}_i \tau^k = \sum_k \sum_i \lambda'_i \phi^*_a(z'_i)^{q^k} \tau^k = \sum_k \sum_i \lambda'_i \left(a_j z'_i\right)^{q^{k-j}} \tau^k \\ &= \sum_k \sum_j \left(\sum_i \lambda'_i \tau^{k-j} z'_i\right) a_j \tau^j = \sum_k \sum_j l'_{k-j} \tau^{k-j} a_j \tau^j = \log'_\phi \circ \phi_a. \end{aligned}$$

Since $\log_\phi$ has the same property, $\log_\phi - \log'_\phi$ is a series in $\mathbb{C}_\infty[[\tau^{-1}]]$ such that $a(\log_\phi - \log'_\phi) = (\log_\phi - \log'_\phi)\phi_a$ for all $a \in A$. If by contradiction $\log_\phi - \log'_\phi \neq 0$, the degrees of both sides would differ for all $a \in A \setminus \mathbb{F}_q$: we deduce that $\log_\phi = \log'_\phi$, hence $l'_k = 0$ for all $k < 0$ and $l'_0 = 1$.

$\square$

5.4 Pairing of Anderson eigenvectors and dual Anderson eigenvectors

Let's fix a Drinfeld module $(\mathbb{G}_a, \phi)$ of rank r.

By Theorem 4.3.32, if $\infty \in X(\mathbb{F}_q)$ and ϕ has rank 1 and is normalized with respect to some sign function $\mathrm{sgn} : A \to \mathbb{F}_q$, the product of an element in $\mathrm{Sf}_{\phi^*}(A)$ and an element in $\mathrm{Sf}_\phi(A)$ is a rational function over $X_{\mathbb{C}_\infty}$.

To generalize this statement to Drinfeld modules of arbitrary rank, we need a proper way of "multiplying" ζ_ϕ and ω_ϕ. The aim of this section is to define a the *dot product* $\zeta_\phi \cdot \omega_\phi \in \mathbb{C}_\infty \hat{\otimes} \Omega$ and study its properties.

5.4.1 Definition and rationality of the dot products $\zeta_\phi \cdot \omega_\phi^{(k)}$

Let's start with the following lemma.

Lemma 5.4.1. *Let $(\lambda_i)_i$ be an $\mathbb{F}_q$-linear basis of Λ_ϕ and $(\lambda_i^*)_i$ the corresponding dual basis of $\hat{\Lambda}_\phi$ The following $A_{\mathbb{C}_\infty}$-linear pairing is well defined:*

$$\begin{array}{ccccc} \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi & \otimes & \mathbb{C}_\infty \hat{\otimes} (\mathrm{Hom}_A(\Lambda_\phi, \Omega)) & \longrightarrow & \mathbb{C}_\infty \hat{\otimes} \Omega \\ \sum_i c_i \otimes \lambda_i & \otimes & \sum_j d_j \otimes (\lambda_j^* \otimes \omega_j) & \mapsto & \sum_{i,j} (c_i d_j) \otimes (\lambda_j^*(\lambda_i) \omega_j) \\ & & \shortparallel & & \shortparallel \\ & & f & & g \end{array}$$

Moreover, under the identifications

$$\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi = \mathrm{Hom}_{\mathbb{F}_q}^{cont} \left(\Lambda_\phi \otimes_A {}^{K_\infty}\!/_{A}, \mathbb{C}_\infty \right) \text{ and } \mathbb{C}_\infty \hat{\otimes} \Omega = \mathrm{Hom}_{\mathbb{F}_q}^{cont} \left({}^{K_\infty}\!/_{A}, \mathbb{C}_\infty \right),$$

for all $b \in {}^{K_\infty}\!/_{A}$ we have:

$$g(b) = \sum_i c_i f(\lambda_i \otimes b).$$

Proof. The morphism is well defined because for all $\varepsilon > 0$ there are finitely many pairs of indices (i,j) such that $\|c_i d_j\| > \varepsilon$; the $A_{\mathbb{C}_\infty}$-linearity is also obvious from the definition. Call $\mathrm{res} : \Omega \otimes K_\infty / A \to \mathbb{F}_q$ and $\mathrm{res}_{\Lambda_\phi} : (\mathrm{Hom}_A(\Lambda_\phi, \Omega)) \otimes \left(\Lambda_\phi \otimes_A {}^{K_\infty}\!/_{A} \right) \to \mathbb{F}_q$ the two perfect pairings outlined in Theorem 2.1.10 and Remark 2.1.11. We have:

$$g(b) = \sum_{i,j} c_i d_j \,\mathrm{res}(\lambda_j^*(\lambda_i) \omega_j, b) = \sum_i c_i \sum_j d_j \,\mathrm{res}_{\Lambda_\phi}(\lambda_j^* \otimes \omega_j, \lambda_i \otimes b) = \sum_i c_i f(\lambda_i \otimes b). \qquad \square$$

The pairing defined in Lemma 5.4.1 is denoted by a dot product. For any element $h \in \mathbb{C}_\infty \hat{\otimes} \Omega = \mathrm{Hom}_{\mathbb{F}_q}^{cont} \left({}^{K_\infty}\!/_{A}, \mathbb{C}_\infty \right)$ and for any $b \in K_\infty$ with projection $\bar{b} \in {}^{K_\infty}\!/_{A}$, to simplify notation we write $h(b)$ to denote $h(\bar{b})$. We can now state a partial generalization of Theorem 4.3.32.

Theorem 5.4.2. *For any Drinfeld module ϕ, for all integers k, the dot product $\zeta_\phi \cdot \omega_\phi^{(k)}$ in $\mathbb{C}_\infty \hat{\otimes} \Omega$ is a rational differential form over the base-changed curve $X_{\mathbb{C}_\infty}$. Moreover, for all positive integers k, $\zeta_\phi \cdot \omega_\phi^{(k)} \in \Omega_{\mathbb{C}_\infty} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} \Omega$.*

Proof. As an element of $\mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({}^{K_\infty\Lambda_\phi}\!/_{\Lambda_\phi}, \mathbb{C}_\infty\right)$, ω_ϕ sends the projection of any $c \in K_\infty\Lambda_\phi$ to $\exp_\phi(c)$ by Theorem 2.2.9. By Lemma 5.4.1, since $\zeta_\phi = -\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \lambda^{-1}\otimes\lambda$, for all $b \in K_\infty$ and for all integers k we have:

$$\zeta_\phi \cdot \omega_\phi^{(k)}(b) = -\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\exp(b\lambda)^{q^k}}{\lambda} = \left(-\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\exp(b\lambda)}{\lambda^{q^{-k}}}\right)^{q^k}.$$

By Proposition 5.1.19, for all positive integers k, if $b \in K_\infty$ has norm at most $q^{-\frac{k+1}{r}}$, $\zeta_\phi \cdot \omega_\phi^{(k)}(b) = 0$. Let's denote by $C \subseteq {}^{K_\infty}\!/_A$ the subspace generated by the projections of elements in K_∞ with norm at most $q^{-\frac{k+1}{r}}$, and denote by Q the quotient ${}^{K_\infty}\!/_A/C$. Since Q is a finite $\mathbb{F}_q$-vector space, we get the following:

$$\mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({}^{K_\infty}\!/_A, \mathbb{C}_\infty\right) \supseteq \mathrm{Hom}^{cont}_{\mathbb{F}_q}(Q, \mathbb{C}_\infty) = \mathrm{Hom}_{\mathbb{F}_q}(Q, \mathbb{C}_\infty) = \mathbb{C}_\infty \otimes \hat{Q}.$$

Since $\zeta_\phi \cdot \omega_\phi^{(k)}$ restricted to C is identically 0, it's contained in $\mathbb{C}_\infty \otimes \hat{Q}$, therefore it can be expressed as a finite sum:

$$\zeta_\phi \cdot \omega_\phi^{(k)} = \sum_i c_i \otimes \mu_i \in \mathbb{C}_\infty \otimes \hat{Q} \subseteq \mathbb{C}_\infty \otimes \widehat{{}^{K_\infty}\!/_A} = \Omega_{\mathbb{C}_\infty}.$$

Since the $\zeta_\phi \cdot \omega_\phi^{(i)}$ is a rational differential form over $X_{\mathbb{C}_\infty}$ for all $i > 0$ we proceed by (backwards) induction to prove it for all negative integers. Fix some $k \le 0$ and suppose that $\zeta_\phi \cdot \omega_\phi^{(i)}$ is a rational differential form over $X_{\mathbb{C}_\infty}$ for all $i > k$; fix some $a \in A \setminus \mathbb{F}_q$. From the definition of special functions we have:

$$\begin{aligned}(1\otimes a - a\otimes 1)\omega_\phi &= \sum_{i=1}^{r\deg(a)} (\phi_a)_i \omega_\phi^{(i)} \\ \Longrightarrow \zeta_\phi \cdot \omega_\phi^{(k)} &= \frac{1}{1\otimes a - a^{q^k}\otimes 1} \sum_{i=1}^{r\deg(a)} (\phi_a)_i^{q^k} \zeta_\phi \cdot \omega_\phi^{(k+i)},\end{aligned}$$

hence $\zeta_\phi \cdot \omega_\phi^{(k)}$ is a rational differential form over $X_{\mathbb{C}_\infty}$. □

Remark 5.4.3. From the previous proof we deduce that, if we can compute the dot product $\zeta_\phi \cdot \omega_\phi^{(k)}$ for $r\deg(a)$ consecutive integers k, then we can compute it for any value of k.

5.4.2 Computation of the dot products $\zeta_\phi \cdot \omega_\phi^{(k)}$ for $k \ll 0$

We can expand on the previous theorem. In fact, we are able to describe explicitly the differential form $\zeta_\phi^{(k)} \cdot \omega_\phi$ for k large enough by using once again Proposition 5.1.19.

Theorem 5.4.4. *For all $b \in K_\infty$ denote by $s(b) \in K_\infty$ an element of smallest norm such that $b - s(b) \in A$. For all integers $k > re\left(\left\lfloor \frac{2g-2}{e} \right\rfloor + 1\right)$, we have the following identity for all $b \in K_\infty$:*

$$\zeta_\phi^{(k)} \cdot \omega_\phi(b) = \sum_{j=0}^{k} e_j l_{k-j}^{q^j} s(b)^{q^j}.$$

Proof. Recall that the norm of all elements in K_∞ is an integer power of q^e. Fix any $b \in K_\infty$, suppose $\|s(b)\| = q^{ed}$ for some integer d; the $\mathbb{F}_q$-vector space $H^0(X, d\infty) / H^0(X, (d-1)\infty)$ has dimension less than e, otherwise there would be some $a \in H^0(X, d\infty) \subseteq A$ such that $\|s(b) - a\| < \|s(b)\|$, contradicting the minimality condition on $s(b)$. By Riemann–Roch, if $e(d-1) > 2g-2$, the spaces $H^0(X, (d-1)\infty)$ and $H^0(X, d\infty)$ have dimension respectively $e(d-1) - g + 1$ and $ed - g + 1$, which is a contradiction, hence $\|s(b)\| \leq q^{e(\lfloor \frac{2g-2}{e} \rfloor + 1)}$. Since $\frac{k-1}{r} \geq e\left(\left\lfloor \frac{2g-2}{e} \right\rfloor + 1\right)$, by Proposition 5.1.19 we have:

$$\zeta_\phi^{(k)} \cdot \omega_\phi(b) = \zeta_\phi^{(k)} \cdot \omega_\phi(s(b)) = -\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{\exp_\phi(s(b)\lambda)}{\lambda^{q^k}} = \sum_{j=0}^{k} e_j l_{k-j}^{q^j} s(b)^{q^j}. \qquad \square$$

Remark 5.4.5. Equivalently, for all integers $k > re\left(\left\lfloor \frac{2g-2}{e} \right\rfloor + 1\right)$ and for all $b \in K_\infty$:

$$\zeta_\phi \cdot \omega_\phi^{(-k)}(b) = \left(\sum_{j=0}^{k} e_j l_{k-j}^{q^j} s(b)^{q^j} \right)^{q^{-k}}.$$

Given any integer i and any $b \in K_\infty$, this result allows us to compute $\zeta_\phi \cdot \omega_\phi^{(i)}(b)$ in the same way we proved rationality in Theorem 5.4.2, as we observed in Remark 5.4.3.

5.4.3 The generating series of the dot products $\zeta_\phi \cdot \omega_\phi^{(k)}$

Using Theorem 5.4.4 and Remark 5.4.5, we can compute the dot product $\zeta_\phi \cdot \omega_\phi^{(k)}$ for any $k \geq -re\left(\left\lfloor \frac{2g-2}{e} \right\rfloor + 1\right)$, but since the sketched algorithm is recursive, it's necessary to compute all the intermediate dot products $\zeta_\phi \cdot \omega_\phi^{(i)}$ for $-re\left(\left\lfloor \frac{2g-2}{e} \right\rfloor + 1\right) \leq i \leq k$.

The objective of this subsection is to streamline this computation by studying the generating series:

$$\sum_{k \in \mathbb{Z}} \zeta_\phi \cdot \omega_\phi^{(k)} \tau^k \in \mathbb{C}_\infty[[\tau, \tau^{-1}]].$$

Definition 5.4.6. Denote by $\mathbb{C}_\infty\langle\tau\rangle$ the subset of $\mathbb{C}_\infty[[\tau]][\tau^{-1}]$ given by the series with a nonzero radius of convergence on $\mathbb{C}_\infty$.

Remark 5.4.7. The set $\mathbb{C}_\infty\langle\tau\rangle$ is closed under addition and composition, hence it is a subring of $\mathbb{C}_\infty[[\tau]][\tau^{-1}]$.

Remark 5.4.8. Since the radius of convergence of $h = \sum_i h_i \tau^i \in \mathbb{C}_\infty[[\tau]][\tau^{-1}]$ is the inverse of $\limsup_{i\to\infty} \|h_i\|^{q^{-i}}$, we have that $h \in \mathbb{C}_\infty\langle\tau\rangle$ if and only if $\limsup_{i\to\infty} \|h_i\|^{q^{-i}} < \infty$.

Lemma 5.4.9. *Every nonzero element $h \in \mathbb{C}_\infty[\tau, \tau^{-1}]$ admits a (unique) bilateral inverse in $\mathbb{C}_\infty\langle\tau\rangle$.*

Proof. There is a unique nonzero $c \in \mathbb{C}_\infty$ and a unique $k \in \mathbb{Z}$ such that $h' := \tau^k c h$ can be written as $\sum_{i\geq 0} h_i \tau^i$ with $h_0 = 1$; since $\tau^k c$ is invertible in $\mathbb{C}_\infty\langle\tau\rangle$, it suffices to prove it for h'. If we call $h_+ := \sum_{i\geq 1} h_i \tau^i$, the series $\sum_{i\geq 0} h_+^i$ is a well defined bilateral inverse of h' in $\mathbb{C}_\infty[[\tau]]$. Since h' has finitely many nonzero coefficients, it's easy to see that there is some $R \in \mathbb{R}_{>0}$ and some positive real constant $C < 1$ such that, for all $x \in \mathbb{C}_\infty$ with norm less than R, $\|h_i x^{q^i}\| \leq C\|x\|$ for all $i \geq 1$. In particular, for all $x \in \mathbb{C}_\infty$ with norm less than R, each of the finitely many summands in the expansion of $h_+^i(x)$ has norm at most $C^i\|x\|$, hence the series $\sum_{i\geq 0} h_+^i(x)$ converges. We deduce that the series $\sum_{i\geq 0} h_+^i$ has a nonzero radius of convergence, hence it belongs to $\mathbb{C}_\infty\langle\tau\rangle$. $\square$

Definition 5.4.10. For all $c \in K_\infty$ we define $\Phi_c \in \mathbb{C}_\infty\langle\tau\rangle$ as $\exp_\phi \circ c \circ \log_\phi$.

Remark 5.4.11. For all $a \in A$, $\Phi_a = \phi_a$. The map $\Phi : K_\infty \to \mathbb{C}_\infty\langle\tau\rangle$ sending c to Φ_c is the unique ring homomorphism which extends $\phi : A \to \mathbb{C}_\infty\langle\tau\rangle$ such that the k-th coefficient $(\Phi)_k : K_\infty \to \mathbb{C}_\infty$ is a continuous function for all $k \in \mathbb{Z}$.

Proposition 5.4.12. *Let $\mu : K_\infty \to \mathbb{C}_\infty[[\tau, \tau^{-1}]]$ be a function with the following properties:*

(i) $\forall k \in \mathbb{Z}$ the function sending c to $(\mu_c)_k$ is $\mathbb{F}_q$-linear and continuous;

(ii) $\forall a \in A, c \in K_\infty$, $\mu_{ac} = \mu_c \phi_a$;

(iii) $\forall a \in A$, $\mu_a = 0$;

(iv) $\forall R \in \mathbb{R} \exists n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$, for all $c \in K_\infty$ with $\|c\| \leq R$, $(\mu_c)_n = (\Phi_c)_n$.

Then, μ is uniquely determined; in particular, for any $c \in K_\infty$, we have:

$$\mu_c = \sum_{k \in \mathbb{Z}} \left(\zeta_\phi^{(k)} \cdot \omega_\phi\right)(c)\tau^k.$$

Proof. To prove uniqueness, let's take two such functions μ and μ', and define $\lambda := \mu - \mu'$. For each element $c \in K_\infty$ let $s(c)$ be an element of least norm such that $c - s(c) \in A$. As we already said in the proof of Theorem 5.4.4, $\|s(c)\| \leq q^{e\left(\lfloor \frac{2g-2}{e} \rfloor + 1\right)}$ for all $c \in K_\infty$; using properties (i),(iii), and (iv) with $R = q^{e\left(\lfloor \frac{2g-2}{e} \rfloor + 1\right)}$, we deduce that there is some integer n_0 such that, for all $n \geq n_0$, for all $c \in K_\infty$:

$$(\lambda_c)_n = (\lambda_{s(c)})_n + (\lambda_{c-s(c)})_n = (\lambda_{s(c)})_n = (\mu_{s(c)})_n - (\mu'_{s(c)})_n = (\Phi_{s(c)})_n - (\Phi_{s(c)})_n = 0.$$

If by contradiction $\lambda \not\equiv 0$, there is an element $c \in K_\infty$ such that λ_c has the highest degree; by property (ii), for any $a \in A \setminus \mathbb{F}_q$, $\lambda_{ac} = \lambda_c \phi_a$, which has a greater degree than λ_c, reaching a contradiction.

Let's check that $\mu_c := \sum_{k \in \mathbb{Z}} \left(\zeta_\phi^{(k)} \cdot \omega_\phi\right)(c)\tau^k$ satisfies the conditions (i)-(iv). The properties (i) and (iii) are obvious. For property (iv), note that for all $c \in K_\infty$

$$(\Phi_c)_k = (\exp_\phi \circ c \circ \log_\phi)_k = \sum_{i+j=k} e_i c^{q^i} l_j^{q^i},$$

which is equal to $\left(\zeta_\phi^{(k)} \cdot \omega_\phi\right)(c)$ for all $k \geq r \cdot \log_q(\|c\|) + 1$ by Proposition 5.1.19. Finally, for property (ii), since ζ_ϕ is an Anderson eigenvector, for all $a \in A$ we have:

$$(1 \otimes a)\zeta_\phi = \sum_{i=0}^{r \deg(a)} (\phi_a)_i^{q^{-i}} \zeta_\phi^{(-i)} \implies \forall k \in \mathbb{Z} : \ (1 \otimes a)\zeta_\phi^{(k)} = \sum_{i=0}^{r \deg(a)} (\phi_a)_i^{q^{k-i}} \zeta_\phi^{(k-i)}.$$

We deduce that, for all $c \in K_\infty$:

$$\begin{aligned}
\mu_c \phi_a &= \left(\sum_{k \in \mathbb{Z}} \left(\zeta_\phi^{(k)} \cdot \omega_\phi\right)(c)\tau^k\right)\left(\sum_{i=0}^{r \deg(a)} (\phi_a)_i \tau^i\right) \\
&= \sum_{k \in \mathbb{Z}} \left(\sum_{i=0}^{r \deg(a)} (\phi_a)_i^{q^{k-i}} \left(\zeta_\phi^{(k-i)} \cdot \omega_\phi\right)(c)\right)\tau^k \\
&= \sum_{k \in \mathbb{Z}} \left((1 \otimes a)\zeta_\phi^{(k)} \cdot \omega_\phi\right)(c)\tau^k = \sum_{k \in \mathbb{Z}} \left(\zeta_\phi^{(k)} \cdot \omega_\phi\right)(ac)\tau^k = \mu_{ac}.
\end{aligned}$$

□

Remark 5.4.13. As a function, μ_c never converges if $c \notin A$. For example, for all $c \in K \setminus A$ we can choose $a \in A$ so that $ca \in A$, and we get that $\mu_c \phi_a = \mu_{ac} = 0$: since $\mu_c \neq 0$, this implies that its radius of convergence is 0.

Definition 5.4.14. For all $c \in K_\infty$ we define $\hat{\Phi}_c \coloneqq (\Phi_c - \mu_c)^* \in \mathbb{C}_\infty[[\tau, \tau^{-1}]]$.

Proposition 5.4.15. *For all $c \in K_\infty$, the series $\hat{\Phi}_c$ has a nonzero radius of convergence. Moreover, the map $\hat{\Phi} : K_\infty \to \mathbb{C}_\infty\langle\tau\rangle$ sending c to $\hat{\Phi}_c$ is the unique ring homomorphism which extends $\phi^* : A \to \mathbb{C}_\infty\langle\tau\rangle$ such that the k-th coefficient $\left(\hat{\Phi}\right)_k : K_\infty \to \mathbb{C}_\infty$ is a continuous function for all $k \in \mathbb{Z}$.*

Proof. Uniqueness is obvious: by multiplicativity there is at most one way to extend ϕ^* to the fraction field K, and by continuity there is at most one way to extend it to the completion K_∞. By definition of Φ and μ, each coefficient of $\hat{\Phi}_c$ is a continuous function of c.

For all $c \in K_\infty$, by Proposition 5.1.19 we have $(\hat{\Phi}_c)_k = ((\Phi_c - \mu_c)^*)_k = 0$ for $k \ll 0$, hence $\hat{\Phi}_c \in \mathbb{C}_\infty[[\tau]][\tau^{-1}]$. On the other hand, for $k \gg 0$:

$$((\hat{\Phi}_c)_k)^{q^{-k}} = ((-\mu_c^*)_k)^{q^{-k}} = -(\mu_c)_{-k} = -\left(\zeta_\phi^{(-k)} \cdot \omega_\phi\right)(c) = \sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{\exp(c\lambda)}{\lambda^{q^{-k}}};$$

all the numerators of the series belong to the compact space $\exp(K_\infty \Lambda_\phi) \cong {}^{K_\infty \Lambda_\phi}\!/_{\Lambda_\phi}$, and since $\Lambda_\phi \subseteq \mathbb{C}_\infty$ is discrete all the denominators are bounded from below: this means that the set $\{((\hat{\Phi}_c)_k)^{q^{-k}}\}_{k \gg 0}$ is bounded, hence $\hat{\Phi}_c \in \mathbb{C}_\infty\langle\tau\rangle$ by Remark 5.4.8. For all $a \in A$, for all $c \in K_\infty$:

$$\hat{\Phi}_a = (\Phi_a - \mu_a)^* = \phi_a^*$$
$$\phi_a^* \circ \hat{\Phi}_c = (\Phi_c \circ \phi_a - \mu_c \circ \phi_a)^* = (\Phi_{ac} - \mu_{ac})^* = \hat{\Phi}_{ac},$$

which proves that $\hat{\Phi}$ extends ϕ^* multiplicatively. □

Remark 5.4.16. For all $c \in K_\infty$ we have:

$$\mu_c^* = \sum_{k \in \mathbb{Z}} \left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(c)\tau^k.$$

In retrospect, we can express the results of this subsection with the following theorem.

Theorem 5.4.17. *Let $\Phi, \hat{\Phi} : K_\infty \to \mathbb{C}_\infty\langle\tau\rangle$ be the unique ring homomorphisms which extend respectively $\phi, \phi^* : A \to \mathbb{C}_\infty\langle\tau\rangle$ and such that their k-th coefficient is a continuous function from K_∞ to $\mathbb{C}_\infty$ for all $k \in \mathbb{Z}$. The following identity holds in the $\mathbb{C}_\infty[\tau, \tau^{-1}]$-module $\mathbb{C}_\infty[[\tau, \tau^{-1}]]$ for all $c \in K_\infty$:*

$$\sum_{k \in \mathbb{Z}} \left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(c)\tau^k = \Phi_c^* - \hat{\Phi}_c.$$

This Theorem allows us to partially carry out the computation of the dot products $\zeta_\phi \cdot \omega_\phi^{(k)}$, such as in the following Proposition.

Proposition 5.4.18. *For all $c \in K_\infty$ with norm less than 1:*

$$\left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(c) = \begin{cases} c \text{ if } k = 0; \\ 0 \text{ if } 1 \le k \le er - 1. \end{cases}$$

Proof. For all $c \in K_\infty$ the lowest degree of $\hat{\Phi}_c$ is $-r\deg(c)$, while the highest degree of Φ_c^* is 0. In particular, if $\|c\| < 1$, i.e. $\deg(c) \leq -e$, we have:

$$\left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(c) = (\Phi_c^* - \hat{\Phi}_c)_k = \begin{cases} (\Phi_c^* - \hat{\Phi}_c)_0 = (\Phi_c^*)_0 = c & \text{if } k = 0; \\ (\Phi_c^* - \hat{\Phi}_c)_k = 0 & \text{if } 1 \leq k \leq er - 1. \end{cases}$$

□

5.5 Explicit computations in some relevant special cases

5.5.1 Application to the case of genus 0 and arbitrary rank

Thanks to Theorem 5.4.17, we can compute efficiently the dot products $\zeta_\phi \cdot \omega_\phi^{(k)}$ in the case of genus 0 and rational point at infinity. In this subsection we suppose $X = \mathbb{P}^1_{\mathbb{F}_q}$, and we fix a rational function θ over X with a simple pole at ∞. In this case we can write $A = \mathbb{F}_q[\theta]$, $K_\infty = \mathbb{F}_q((\theta^{-1}))$ and $\Omega = \mathbb{F}_q[\theta]\mathrm{d}\theta$, where $\mathrm{d}\theta : {K_\infty}/{A} \to \mathbb{F}_q$ sends θ^n to $\delta_{-1,n}$ for all $n \in \mathbb{Z}$.

Proposition 5.5.1. *Let $\phi : \mathbb{F}_q[\theta] \to \mathbb{C}_\infty[\tau]$ be a Drinfeld module of rank r. We have the following identities in $\mathbb{C}_\infty \hat{\otimes} \Omega$:*

$$\zeta_\phi \cdot \omega_\phi = \frac{\mathrm{d}\theta}{\theta \otimes 1 - 1 \otimes \theta};$$
$$\zeta_\phi \cdot \omega_\phi^{(k)} = 0 \quad \forall 1 \leq k \leq r-1.$$

Proof. By Proposition 5.4.18, for all $n > 0$ we have:

$$\left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(\theta^{-n}) = \begin{cases} \theta^{-n} \text{ if } k = 0 \\ 0 \text{ if } 1 \leq k \leq r-1 \end{cases}.$$

Since $\theta^n \in A$ for all $n \geq 0$, we also have $\left(\zeta_\phi \cdot \omega_\phi^{(k)}\right)(\theta^n) = 0$ for all $n \geq 0$ and for all k so, if $1 \leq k \leq r-1$, $\zeta_\phi \cdot \omega_\phi^{(k)}$ is identically zero. If instead $k = 0$ we have the following identity for all integers n:

$$((\theta \otimes 1 - 1 \otimes \theta)\zeta_\phi \cdot \omega_\phi)(\theta^n) = \theta(\zeta_\phi \cdot \omega_\phi(\theta^n)) - \zeta_\phi \cdot \omega_\phi(\theta^{n+1}) = \delta_{-1,n} = \mathrm{d}\theta(\theta^n),$$

hence $\zeta_\phi \cdot \omega_\phi = \frac{\mathrm{d}\theta}{(\theta \otimes 1 - 1 \otimes \theta)}$. □

We now relate the usual definition of Anderson generating functions to the universal Anderson eigenvector, by giving a basis-dependent description of the latter.

Lemma 5.5.2. *Fix the A-linear bases $\{\pi_1, \ldots, \pi_r\}$ of Λ_ϕ and $\{\pi_1^*, \ldots, \pi_r^*\}$ of $\mathrm{Hom}_A(\Lambda_\phi, A)$, where $\pi_i^*(\pi_k) = \delta_{i,k}$. Then, we have:*

$$\omega_\phi = \sum_{i=1}^r \sum_{j \geq 0} \exp_\phi\left(\frac{\pi_i}{\theta^{j+1}}\right) \otimes \theta^j \pi_i^* \mathrm{d}\theta, \qquad \zeta_\phi = \sum_{i=1}^r \sum_{j \geq 0} \left(\sum_{\lambda \in \Lambda_\phi \setminus \{0\}} \frac{\mathrm{d}\theta \pi_i^*}{\theta^{j+1}}(\lambda)\lambda^{-1} \right) \otimes \theta^j \pi_i.$$

Proof. When used as indices, we imply i to vary among the integers between 1 and r, extremes included, and j to vary among the nonnegative integers. The chosen bases induce an isomorphism $\mathrm{Hom}_A(\Lambda_\phi, \Omega) \cong \bigoplus_i A\mathrm{d}\theta\pi_i^*$. The $\mathbb{F}_q$-linear basis $\{\theta^j \mathrm{d}\theta\pi_i^*\}_{i,j}$ of $\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ induces a dual basis $\{\theta^{-j-1}\pi_i\}_{i,j}$ of $\widehat{\mathrm{Hom}_A(\Lambda_\phi, \Omega)} \cong {}^{K_\infty\Lambda_\phi}\!/_{\Lambda_\phi}$. Similarly, the $\mathbb{F}_q$-linear basis $\{\theta^j\pi_i\}_{i,j}$ of Λ_ϕ: induces the dual basis $\{\theta^{-j-1}\mathrm{d}\theta\pi_i^*\}_{i,j}$ of $\hat{\Lambda}_\phi \cong {}^{K_\infty \mathrm{Hom}_A(\Lambda_\phi, \Omega)}\!/_{\mathrm{Hom}_A(\Lambda_\phi, \Omega)}$. This proves the lemma, by virtue of Remark 2.2.13 and the proof of Theorem 5.2.10. $\square$

Definition 5.5.3. For $i = 1, \dots, r$ we define the i-th Anderson generating function as:

$$\omega_{\phi,i} := \sum_{j\geq 0} \exp_\phi\left(\frac{\pi_i}{\theta^{j+1}}\right) \otimes \theta^j \in \mathbb{C}_\infty \hat{\otimes} A.$$

Similarly, for $i = 1, \dots, r$ we define the i-th dual Anderson generating function as:

$$\zeta_{\phi,i} = \sum_{j\geq 0} \left(\sum_{\lambda\in\Lambda_\phi\setminus\{0\}} \frac{\mathrm{d}\theta\pi_i^*}{\theta^{j+1}}(\lambda)\lambda^{-1} \right) \otimes \theta^j \in \mathbb{C}_\infty \hat{\otimes} A.$$

Remark 5.5.4. For all integers $1 \leq i \leq r$, $\omega_{\phi,i}$ and $\zeta_{\phi,i}$ are the unique elements in $\mathbb{C}_\infty \hat{\otimes} A$ such that the identities $(1 \otimes \pi_i)(\omega_\phi) = \omega_{\phi,i}\mathrm{d}\theta$ and $(1 \otimes \pi_i^*)(\zeta_\phi) = \zeta_{\phi,i}$ hold (in $\mathbb{C}_\infty \hat{\otimes} \Omega$ and $\mathbb{C}_\infty \hat{\otimes} A$, respectively).

Definition 5.5.5. Let's define $\boldsymbol{\omega}_\phi := (\omega_{\phi,i}^{(j-1)})_{i,j} \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty \hat{\otimes} A)$. We call it the *rigid analitic trivialization* of the t-motive attached to ϕ.

The previous matrix has been studied in various articles (see for example [Pel08][Section 4.2], [KP23], [GP19]). We can use it to state the following Theorem.

Theorem 5.5.6. *The product of $\zeta_\phi \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty \hat{\otimes} A)$ and $\boldsymbol{\omega}_\phi \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty \hat{\otimes} A)$ is the vector $\frac{1}{(\theta\otimes 1 - 1\otimes\theta)} \cdot (1, 0, \dots, 0) \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty \hat{\otimes} A)$.*

Proof. Note that we have interpreted ζ_ϕ as $(\zeta_{\phi,i})_i \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty \hat{\otimes} A)$. If we multiply by $\mathrm{d}\theta \in \Omega$ the j-th coordinate of the product, we get:

$$\sum_{i=1}^r \omega_{\phi,i}^{(j-1)} \zeta_{\phi,i} \mathrm{d}\theta = \left(\sum_{i=1}^r \omega_{\phi,i}\pi_i^* \mathrm{d}\theta\right)^{(j-1)} \cdot \left(\sum_{i=1}^r \zeta_{\phi,i}\pi_i\right) = \omega_\phi^{(j-1)} \cdot \zeta_\phi,$$

which is $\frac{\mathrm{d}\theta}{(\theta\otimes 1 - 1\otimes\theta)}$ if $j = 1$ by Proposition 5.5.1, and 0 otherwise by Proposition 5.4.18. $\square$

Remark 5.5.7. It's a well known result that the determinant of the matrix $\boldsymbol{\omega}_\phi$ is nonzero (see for example [GP19][Prop. 6.2.4]), so by the previous theorem we can recover ζ_ϕ from $\boldsymbol{\omega}_\phi$.

5.5.2 Application to the case of hyperelliptic curves

In the case of rank 1 normalized Drinfeld modules, Theorem 4.3.32 can be used to express the rational form $\zeta_\phi \cdot \omega_\phi$ in terms of the Drinfeld divisor. While Theorem 5.4.17, in principle, completely describes the form $\zeta_\phi \cdot \omega_\phi$, it's not as explicit a result for arbitrary curves.

In this subsection we restrict ourselves to the case of a hyperelliptic curve X with hyperelliptic divisor 2∞ and a Drinfeld module ϕ of rank 1. We use the results of the previous sections to recover an expression for the scalar product $\zeta_\phi \cdot \omega_\phi$ and for the shtuka function f_ϕ in terms of the coefficients of ϕ.

A curve X of genus g is hyperelliptic if and only if there is a divisor D of degree 2, called hyperelliptic divisor, such that $\dim_{\mathbb{F}_q}(H^0(X, D)) = 2$. If we assume $D = 2\infty$, there is a rational function $x \in A$ of degree 2. Let's denote by y an element of A with the smallest odd degree.

Remark 5.5.8. An $\mathbb{F}_q$-linear basis of A is $\mathcal{B}_0 := \{x^i, x^i y\}_{i\geq 0}$. In particular, the only positive integers that are not degrees of elements in A are the odd positive integers smaller than $\deg(y)$; by Riemann–Roch's theorem, this set has cardinality g, hence $\deg(y) = 2g+1$. Expanding y^2 in terms of the basis $\mathcal{B}_0$, we deduce that there are polynomials $P, Q \in \mathbb{F}_q[t]$ such that $y^2 = Q(x)y + P(x)$, where P has degree $2g+1$ and Q has degree at most g.

If the characteristic of the base field is odd, we can also assume $Q(x) = 0$ using the coordinate change $y \mapsto y + \frac{Q(x)}{2}$.

Remark 5.5.9. Every element of ${K_\infty}/{A}$ can be represented by an element of K_∞ with degree either negative or equal to an odd positive number smaller than $2g+1$. We deduce that the image of $\mathcal{B} := \{yx^{-i-1}, x^{-i-1}\}_{i\geq 0}$ in ${K_\infty}/{A}$ is a set of linearly independent elements which spans a dense subset of ${K_\infty}/{A}$.

Proposition 5.5.10. *If we define $\nu \in \Omega = \mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({K_\infty}/{A}, \mathbb{F}_q\right)$ as the function sending yx^{-1} to 1 and all the other elements of $\mathcal{B}$ to 0, we get that $\Omega = A\nu$.*

Proof. For all $j \geq 0$, for all $c \in K_\infty$, $(x^j\nu)(c) = \nu(x^j c)$, which is 1 when $c = yx^{-j-1}$ and 0 on all the other elements of $\mathcal{B}$.

Similarly, For all $j \geq 0$, for all $c \in K_\infty$, $((y - Q(x))x^j\nu)(c) = \nu((y - Q(x))x^j c)$. If $c = x^{-1-i}$ for some $i \geq 0$ we have:

$$((y - Q(x))x^j\nu)(c) = \nu(yx^{j-i-1}) - \nu(Q(x)x^{j-i-1}) = \nu(yx^{j-i-1}) = \delta_{j,i}.$$

If $c = yx^{-1-i}$ for some $i \geq 0$ we have:

$$((y - Q(x))x^j\nu)(c) = \nu((y^2 - Q(x)y)x^{j-i-1}) = \nu(P(x)x^{j-i-1}) = 0.$$

In particular, the elements $\{(y - Q(x))x^i\nu, x^i\nu\}_{i\geq 0} \subseteq \Omega = \mathrm{Hom}_{\mathbb{F}_q}\left({K_\infty}/{A}, \mathbb{F}_q\right)$ are independent, and since $\mathcal{B}$ spans a dense subset of ${K_\infty}/{A}$, they also generate all of Ω. $\square$

Lemma 5.5.11. *Denote by $\left({K_\infty}/{A}\right)_{<q^{-2}} \subseteq {K_\infty}/{A}$ the subspace of the elements with norm less than q^{-2}, and call C the cokernel of this inclusion. Then, the images of $\{yx^{-i-1}\}_{0\leq i\leq g} \cup \{x^{-1}\}$ form a basis of C, and the set $\{x^i\nu\}_{0\leq i\leq g} \cup \{(y - Q(x))\nu\}$ is the corresponding dual basis of $\mathrm{Hom}_{\mathbb{F}_q}(C, \mathbb{F}_q) \subseteq \Omega$.*

Proof. On one hand, the images of $\{yx^{-i-1}\}_{0\leq i\leq g} \cup \{x^{-1}\}$ span C because they are the only elements of $\mathcal{B}$ that are not sent to 0 under the induced map $K_\infty \to C$. On the other hand, $\deg(yx^{-i-1}) = 2(g-i) - 1$ for all $0 \leq i \leq g$, and $\deg(x^{-1}) = -2$, hence their images are $\mathbb{F}_q$-linearly independent in C.

Note that the image of $\{yx^{-j-1-g}, x^{-j-1}\}_{j\geq 1}$ in ${K_\infty}/{A}$ spans a dense subset of $\left({K_\infty}/{A}\right)_{<q^{-2}}$. For all $0 \leq i \leq g$, for all $j \geq 1$ we get:

$$\left.\begin{array}{ll}
(x^i\nu)(yx^{-j-1-g}) = \nu(yx^{i-j-1-g}) = 0 & \text{because } i - j - 1 - g \leq -2 \\
(x^i\nu)(x^{-j-1}) = \nu(x^{i-j}) = 0 & \\
((y - Q(x))\nu)(yx^{-j-1-g}) = \nu(P(x)x^{-j-1-g}) = 0 & \\
((y - Q(x))\nu)(x^{-j-1}) = \nu(yx^{-j-1}) - \nu(Q(x)x^{-j-1}) = 0 & \text{because } -j - 1 \leq -2
\end{array}\right\}$$

so $\{x^i\nu\}_{0\leq i\leq g}\cup\{y\nu\}\in \mathrm{Hom}_{\mathbb{F}_q}(C,\mathbb{F}_q)$. On the other hand, we have the following identities: for all $0\leq i\leq g$ and for all $o\leq j\leq g$:

$$(x^i\nu)(yx^{-j-1})=\nu(yx^{i-j-1})=\delta_{i,j};\qquad ((y-Q(x))\nu)(yx^{-j-1})=\nu(P(x)x^{-j-1})=0;$$
$$(x^i\nu)(x^{-1})=\nu(x^{i-1})=0;\qquad ((y-Q(x))\nu)(x^{-1})=\nu(yx^{-1})-\nu(Q(x)x^{-1})=1.$$

This proves that $\{x^i\nu\}_{0\leq i\leq g}\cup\{(y-Q(x))\nu\}$ is the dual basis of $\{yx^{-i-1}\}_{0\leq i\leq g}\cup\{x^{-1}\}$. □

By Theorem 5.4.17, we have the following identity for all $c\in K$ and for all $i\in\mathbb{Z}$:

$$\left(\zeta_\phi\cdot\omega_\phi^{(i)}\right)(c)=((\phi_c)^*-(\phi^*)_c)_i,$$

where $[\left(\zeta_\phi\cdot\omega_\phi^{(i)}\right)$ is considered as a continuous homomorphism from ${K_\infty}/{A}$ to $\mathbb{C}_\infty$. In particular, for all $c\in K$ with degree less than $-i$:

$$\left(\zeta_\phi\cdot\omega_\phi^{(i)}\right)(c)=\begin{cases}c \text{ if } i=0\\ 0 \text{ if } i>0.\end{cases}$$

Moreover, for all $0\leq i\leq g$ we have:

$$(\zeta_\phi\cdot\omega_\phi)(yx^{-i})=yx^{-i}-\left(\phi^*_{yx^{-i}}\right)_0;$$
$$\left(\zeta_\phi\cdot\omega_\phi^{(1)}\right)(yx^{-i})=-\left(\phi^*_{yx^{-i}}\right)_1.$$

Theorem 5.5.12. *We have the following identities for the dot product $\zeta_\phi\cdot\omega_\phi$ and the shtuka function f_ϕ:*

$$\zeta_\phi\cdot\omega_\phi=\left(\frac{y\otimes1+1\otimes(y-Q(x))}{x\otimes1-1\otimes x}-\sum_{i=0}^{g-1}\left(\phi^*_{yx^{-i-1}}\right)_0\otimes x^i\right)(1\otimes\nu)$$
$$f_\phi=\frac{(x\otimes1-1\otimes x)\left(-\sum_{i=0}^{g}\left(\phi^*_{yx^{-i-1}}\right)_1\otimes x^i\right)}{y\otimes1+1\otimes(y-Q(x))-(x\otimes1-1\otimes x)\left(\sum_{i=0}^{g-1}\left(\phi^*_{yx^{-i-1}}\right)_0\otimes x^i\right)}.$$

Proof. For all $c\in K_\infty$ of norm less than 1, $(\zeta_\phi\cdot\omega_\phi)(c)=c$. In particular, for all $c\in\left({K_\infty}/{A}\right)_{<q^2}$ we have:

$$(x\otimes1-1\otimes x)(\zeta_\phi\cdot\omega_\phi)(c)=x(\zeta_\phi\cdot\omega_\phi)(c)-(\zeta_\phi\cdot\omega_\phi)(xc)=0.$$

In particular, by Lemma 5.5.11 $(x\otimes1-1\otimes x)(\zeta_\phi\cdot\omega_\phi)$ is completely determined by its evaluation at $\{yx^{-i-1}\}_{0\leq i\leq g}\cup\{x^{-1}\}$ as a function from ${K_\infty}/{A}$ to $\mathbb{C}_\infty$. Since $(\zeta_\phi\cdot\omega_\phi)(yx^{-i})=yx^{-i}-\left((\phi_x^*)^{-i}\circ\phi_y^*\right)_0$ for all $0\leq i\leq g$, we can compute the following evaluations:

$$\begin{aligned}(x\otimes1-1\otimes x)(\zeta_\phi\cdot\omega_\phi)(yx^{-i-1})&=x(\zeta_\phi\cdot\omega_\phi)(yx^{-i-1})-(\zeta_\phi\cdot\omega_\phi)(yx^{-i})\\&=\left(\phi^*_{yx^{-i}}-x\phi^*_{yx^{-i-1}}\right)_0;\\(x\otimes1-1\otimes x)(\zeta_\phi\cdot\omega_\phi)(x^{-1})&=x(\zeta_\phi\cdot\omega_\phi)(x^{-1})-(\zeta_\phi\cdot\omega_\phi)(1)=1;\\\left(\zeta_\phi\cdot\omega_\phi^{(1)}\right)(yx^{-i-1})&=-\left(\phi^*_{yx^{-i-1}}\right)_1\\\left(\zeta_\phi\cdot\omega_\phi^{(1)}\right)(x^{-1})&=0.\end{aligned}$$

By Lemma 5.5.11, and using that $\phi^*_{yx^{-g-1}}$ has degree 1 in τ, we deduce the following identities:

$$(x\otimes 1 - 1\otimes x)(\zeta_\phi\cdot\omega_\phi) = \left(\sum_{i=0}^{g}\left(\phi^*_{yx^{-i}} - x\phi^*_{yx^{-i-1}}\right)_0 \otimes x^i + 1\otimes(y-Q(x))\right)(1\otimes\nu)$$
$$= \left(\left(\phi^*_y\right)_0\otimes 1 + \sum_{i=0}^{g-1}\left(\phi^*_{yx^{-i-1}}\right)_0\otimes x^{i+1} - \sum_{i=0}^{g-1}\left(x\phi^*_{yx^{-i-1}}\right)_0\otimes x^i + 1\otimes(y-Q(x))\right)(1\otimes\nu)$$
$$= \left((1\otimes x - x\otimes 1)\left(\sum_{i=0}^{g-1}\left(\phi^*_{yx^{-i-1}}\right)_0\otimes x^i\right) + y\otimes 1 + 1\otimes(y-Q(x))\right)(1\otimes\nu)$$
$$f_\phi = \frac{(x\otimes 1 - 1\otimes x)\left(\zeta_\phi\cdot\omega_\phi^{(1)}\right)}{(x\otimes 1-1\otimes x)(\zeta_\phi\cdot\omega_\phi)}$$
$$= \frac{(x\otimes 1 - 1\otimes x)\left(-\sum_{i=0}^{g}\left(\phi^*_{yx^{-i-1}}\right)_1\otimes x^i\right)}{y\otimes 1 + 1\otimes(y-Q(x)) - (x\otimes 1 - 1\otimes x)\left(\sum_{i=0}^{g-1}\left(\phi^*_{yx^{-i-1}}\right)_0\otimes x^i\right)}. \qquad \square$$

5.5.3 Comparison with known results in the case of elliptic curves

The computations can be directly compared to the results of Green and Papanikolas, who tackled the particular case of an elliptic curve in [GP18]. They assumed ϕ to be normalized and the period lattice Λ_ϕ to be isomorphic to A, and they set:

$$\phi_x = x + x_1\tau + \tau^2 \qquad\qquad \phi_y = y + y_1\tau + y_2\tau^2 + \tau^3.$$

They proved the following identities (see [GP18][Thm. 7.1, Eqq. 18,26,27]):

$$f_\phi = \frac{1\otimes y - y\otimes 1 - ((y_2 - x_1^q)\otimes 1)(1\otimes x - x\otimes 1)}{1\otimes x - x^q\otimes 1 + (y_1 - x_1(y_2-x_1^q))\otimes 1};$$
$$\zeta_\phi\cdot\omega_\phi = \frac{(x^q - y_1 + x_1(y_2-x_1^q))^q\otimes 1 - 1\otimes x}{f_\phi}.$$

Let's compare these results with Theorem 5.5.12. First, we need to compute the coefficients $(\phi^*_{yx^{-1}})_0$, $(\phi^*_{yx^{-1}})_1$, $(\phi^*_{yx^{-2}})_1$. Starting from the definition of ϕ^*_x and ϕ^*_y we can explicitly compute the first 3 terms of $\phi^*_{yx^{-1}}$ using the identity $\phi^*_x\phi^*_{yx^{-1}} = \phi^*_y$:

$$\phi^*_x = \tau^{-2} + x_1^{q^{-1}}\tau^{-1} + x$$
$$\phi^*_y = \tau^{-3} + y_2^{q^{-2}}\tau^{-2} + y_1^{q^{-1}}\tau^{-1} + y$$
$$\phi^*_{yx^{-1}} = \tau^{-1} + (y_2 - x_1^q) + (y_1^q - x_1^q y_2^q - x^{q^2} + x_1^{q^2+q})\tau + \dots,$$

hence $(\phi^*_{yx^{-1}})_0 = y_2 - x_1^q$ and $(\phi^*_{yx^{-1}})_1 = y_1^q - x_1^q y_2^q - x^{q^2} + x_1^{q^2+q}$. Since $\deg(yx^{-2}) = -1$, and since ϕ is normalized, we have $\phi^*_{yx^{-2}} \in \tau + \mathbb{C}_\infty[[\tau]]\tau^2$, hence $(\phi^*_{yx^{-2}})_1 = 1$. By Theorem 5.5.12, we have:

$$(\zeta_\phi\cdot\omega_\phi)f_\phi = (\zeta_\phi\cdot\omega_\phi^{(1)}) = -\sum_{i=0}^{g}\left(\phi^*_{yx^{-i-1}}\right)_1\otimes x^i = -((y_1^q - x_1^q y_2^q - x^{q^2} + x_1^{q^2+q})\otimes 1 + 1\otimes x),$$

which agrees with Green and Papanikolas' formula for $\zeta_\phi\cdot\omega_\phi$.

Remark 5.5.13. In retrospect, since the computations do not take into account the A-module structure of Λ_ϕ, it turns out that the formulas found by Green and Papanikolas hold without the assumption $\Lambda_\phi \cong A$.

Chapter 6

Approach to a generalization for abelian Anderson A-modules

One of the purposes of this last chapter is to draw some comparisons between the theory developed in this thesis and the article [HJ20] by Hartl and Juschka, where they explore the relation between Anderson motives and dual Anderson motives (whose field of definition is assumed to be $\mathbb{C}_\infty$). Furthermore, we formulate several conjectural generalizations for the theorems proven in this thesis to uniformizable abelian Anderson A-modules. Finally, some of these generalizations are proven in the special case of the tensor power of the Carlitz module.

6.1 Anderson A-motives

For all $f \in A_{\mathbb{C}_\infty} := \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$, denote by $f^{(1)}$ the image of f under the Frobenius twist. We define $A_{\mathbb{C}_\infty}[\tau]$ as the noncommutative $A_{\mathbb{C}_\infty}$-algebra generated by τ with the relation $\tau \cdot f = f^{(1)} \cdot \tau$ for all $f \in A_{\mathbb{C}_\infty}$.

Definition 6.1.1. Let M be a left $A_{\mathbb{C}_\infty}[\tau]$-module with the following properties:

- M is projective of finite rank as an $A_{\mathbb{C}_\infty}$-module;
- for all $a \in A$ there is some positive integer n such that $(a \otimes 1 - 1 \otimes a)^n \cdot M \subseteq \tau \cdot M$.

Then, M is called an *effective Anderson A-motive*.

Effective Anderson $\mathbb{F}_q[t]$-motives were originally called "t-motives", and were introduced by Anderson in his seminal paper [And86], under the hypothesis $A = \mathbb{F}_q[t]$, to answer some open questions, for example about the uniformizability of Anderson A-modules.

Our naming convention follows the comprehensive work of Hartl and Juschka ([HJ20]), whose definition of Anderson A-motives is slightly more general ([HJ20][Def. 2.3.1]), but superfluous for the sake of this chapter.

The most important feature of Anderson A-motives is that they seem to play the same role as Grothendieck motives for algebraic varieties—which are meant to work as a “universal cohomology theory"—while being concrete, as they form an explicit subcategory of the category of projective $A_{\mathbb{C}_\infty}$-modules. For example, while there is no generally accepted construction of a $\mathbb{Q}$-linear category of mixed Grothendieck motives, and its existence is entirely conjectural, the category of mixed A-motives is explicit and well understood (see for example [HJ20][Section 2.3]).

Interestingly, while the Hodge conjecture is a longstanding open problem in the context of complex varieties, the analogue of the Hodge conjecture has been proven to hold by Pink and Hartl for the category of mixed uniformizable A-motives (see [HJ20]).

Definition 6.1.2. Let N be a right $A_{\mathbb{C}_\infty}[\tau]$-module with the following properties:

- N is projective of finite rank as an $A_{\mathbb{C}_\infty}$-module;
- for all $a \in A$ there is some positive integer n such that $N \cdot (a \otimes 1 - 1 \otimes a)^n \subseteq N \cdot \tau$.

Then, N is called an *effective dual Anderson A-motive.*

Effective dual $\mathbb{F}_q[t]$-motives were first introduced in [ABP04] (where they were called "dual t-motives"), and were used to give a theoretical framework for a well-known linear independence criterion over function fields ([ABP04][Thm. 1.3.2]).

With the following definitions we establish a link between Anderson A-modules and (dual) A-motives. Given two affine $\mathbb{F}_q$-module schemes G, G' over $\mathbb{C}_\infty$, let's denote by $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(G, G')$ the set of morphisms from G to G'.

Remark 6.1.3. Note that $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(\mathbb{G}_a, \mathbb{G}_a) = \mathbb{C}_\infty[\tau]$. On one hand, $\mathbb{C}_\infty[\tau]$ acts by post-composition on $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(E, \mathbb{G}_a)$; on the other hand, A acts by pre-composition via ϕ. The two actions endow $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(E, \mathbb{G}_a)$ with a natural structure of left $A_{\mathbb{C}_\infty}[\tau]$-module.

Similarly, $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(\mathbb{G}_a, E)$ can be endowed with a natural structure of right $A_{\mathbb{C}_\infty}[\tau]$-module.

Definition 6.1.4. Let $\underline{E} = (E, \phi)$ be an Anderson A-module and suppose that $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(E, \mathbb{G}_a)$ is an effective A-motive. In that case, $\underline{E}$ is said to be *abelian*, and we denote its associated A-motive $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(E, \mathbb{G}_a)$ as $M(\underline{E})$.

Definition 6.1.5. Let $\underline{E} = (E, \phi)$ be an Anderson A-module and suppose that $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(\mathbb{G}_a, E)$ is an effective dual A-motive. In that case, $\underline{E}$ is said to be *A-finite*, and we denote its associated dual A-motive $\mathrm{Hom}_{\mathbb{F}_q,\mathbb{C}_\infty}(\mathbb{G}_a, E)$ as $N(\underline{E})$.

The correspondence $\underline{E} \mapsto M(\underline{E})$ is an antiequivalence between the category of abelian Anderson A-modules and that of effective A-motives ([Gos98][Thm. 5.4.11]). In the preprint [Mau24], Maurischat proved that an Anderson A-module $\underline{E}$ is abelian if and only if it is A-finite.

6.2 Link with the Hartl–Juschka pairing

Let $\underline{E}$ be an abelian Anderson A-module, and denote by $\tau M(\underline{E})$ as the image of the endomorphism $\tau : M(\underline{E}) \to M(\underline{E})$; we can also think of it as the pullback of the $A_{\mathbb{C}_\infty}$-module $M(\underline{E})$ along the Frobenius twist $\cdot^{(1)} : A_{\mathbb{C}_\infty} \to A_{\mathbb{C}_\infty}$.

Let $\tilde{N}(\underline{E})$ denote the finite projective $A_{\mathbb{C}_\infty}$-module $\mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \Omega_{\mathbb{C}_\infty})$; we can endow it with a structure of right $A_{\mathbb{C}_\infty}[\tau]$-module by setting $f \cdot \tau$ as the map sending $m \in \tau M(\underline{E})$ to $f(\tau \cdot m)^{(-1)}$ for all $f \in \tilde{N}(\underline{E})$.

In their paper [HJ20], Hartl and Juschka proved the following isomorphism.

Theorem 6.2.1 ([HJ20][Thm. 2.5.13])**.** *Let $\underline{E}$ be an abelian and A-finite Anderson A-module. There is a natural isomorphism of right $A_{\mathbb{C}_\infty}[\tau]$-modules between $N(\underline{E})$ and $\tilde{N}(\underline{E})$.*

In this section, using the dot product defined in Lemma 5.4.1, we give an alternative proof of this theorem as Theorem 6.2.7 in the special case of a Drinfeld module.

In the same generality, we also partially answer the following question by Hartl and Juschka, which they only tackled for $\mathbb{F}_q[t]$-Drinfeld modules.

Question ([HJ20][2.5.15])**.** If $\underline{E}$ is an abelian and A-finite Anderson A-module, the isomorphism from Theorem 6.2.1 defines a perfect pairing of $A_{\mathbb{C}_\infty}$-modules

$$HJ : N(\underline{E}) \otimes_{A_{\mathbb{C}_\infty}} \tau M(\underline{E}) \to \Omega_{\mathbb{C}_\infty}.$$

Is it possible to give a direct description of this pairing?

With Theorem 6.2.7, we argue that, when $\underline{E}$ is a Drinfeld module, Hartl and Juschka's perfect pairing is induced by the dot product defined in Subsection 5.4.1.

6.2.1 The case of Drinfeld A-modules

Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld A-module, so that $\underline{E}$ is abelian (and A-finite) and we can canonically identify the A-motive $M(\underline{E})$ with $\mathbb{C}_\infty[\tau]$ and $\tau M(\underline{E})$ with $\tau\mathbb{C}_\infty[\tau]$.

Proposition 6.2.2. *Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld A-module. There is a natural immersion of left $A_{\mathbb{C}_\infty}[\tau]$-modules from $M(\underline{E})$ to $\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ which induces an isomorphism between $\overline{M(\underline{E})} := M(\underline{E}) \otimes_{A_{\mathbb{C}_\infty}} (\mathbb{C}_\infty \hat{\otimes} A)$ and $\mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$.*

Proof. The natural map of left $A_{\mathbb{C}_\infty}[\tau]$-modules

$$M(\underline{E}) = \operatorname{Hom}_{\mathbb{F}_q, \mathbb{C}_\infty}(E, \mathbb{G}_a) \to \operatorname{Hom}^{cont}_{\mathbb{F}_q}(\exp_\phi(K_\infty \Lambda_\phi), \mathbb{C}_\infty) \cong \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$$

sends a polynomial f to the restriction of the associated continuous function $f(\mathbb{C}_\infty) : E(\mathbb{C}_\infty) \to \mathbb{C}_\infty$ to $\exp_\phi(K_\infty \Lambda_\phi) \subseteq E(\mathbb{C}_\infty)$.

Let's prove that the induced map from $\overline{M(\underline{E})}$ to $\mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$ is bijective, starting with injectivity; since $M(\underline{E})$ is a flat $A_{\mathbb{C}_\infty}$-module, this also proves that the map $M(\underline{E}) \hookrightarrow \overline{M(\underline{E})} \to \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$ is an immersion.

First note that, for any $t \in A \setminus \mathbb{F}_q$, $\overline{M(\underline{E})} = M(\underline{E}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty \hat{\otimes} \mathbb{F}_q[t]$, hence any element $f \in \overline{M(\underline{E})}$ can be written in a unique way as $\sum_{i=0}^{r-1} \left(\sum_{n \geq 0} c_{i,n} \otimes t^n \right) \cdot \tau^i$, where r is the degree of ϕ_t in τ. Suppose that its image $f(\mathbb{C}_\infty) \in \operatorname{Hom}^{cont}_{\mathbb{F}_q}(\exp_\phi(K_\infty \Lambda_\phi), \mathbb{C}_\infty)$ is identically zero. Fix some integer $m \geq 1$: for all $x \in \ker(\phi_{t^m}) \subseteq \exp_\phi(K_\infty \Lambda_\phi)$ we have

$$0 = f(\mathbb{C}_\infty)(x) = \sum_{i=0}^{r-1} \left(\sum_{n=0}^{r-1} c_{i,n} \tau^i \phi_{t^n} \right)(x),$$

and since $\# \ker(\phi_{t^m}) = q^{mr}$ and the polynomial on the right hand side has degree less than q^{mr}, it must be identically zero. Since the polynomials $\{\tau^i \phi_{t^n}\}$ all have different degrees, they are $\mathbb{C}_\infty$-linearly independent, hence $c_{i,n} = 0$ for all $i = 0, \dots, r-1$ and all $n = 0, \dots, m-1$. Finally, since m is arbitrary, we deduce that $f = 0$.

Let's prove surjectivity. Pick some $f : \exp_\phi(K_\infty \Lambda_\phi) \to \mathbb{F}_q$ and call $V := \ker(f)$; fix some $a \in A$ such that $V \cap \ker \phi_a \subsetneq \ker \phi_a$ and define $V_n := V \cap \ker \phi_{a^n}$ for all $n \geq 1$: we know that $V_n \subsetneq \ker \phi_{a^n}$ has codimension 1. Denote by r the degree in τ of ϕ_a and pick some $x_0 \in \ker \phi_a$ of maximal norm such that $f(x_0) = 1$. For all $n \geq 1$, we define $p_n \in \mathbb{C}_\infty[\tau]$ as the unique additive polynomial of degree

$nr-1$ such that $p_n|_{V_n}=0$ and $p_n(x_0)=1$: in particular, $(1-\tau)p_n|_{\ker\phi_{a^n}}=0$, hence there is some $\alpha_n\in\mathbb{C}_\infty$ such that $(1-\tau)p_n=\alpha_n\phi_{a^n}$. Since $p_n(x_0)=1$, the leading term of p_n is $\prod_{x\in V_n}(x_0-x)^{-1}$. On the other hand, the leading term of ϕ_{a^n} is $a^n\cdot\prod_{x\in\ker\phi_{a^n}\setminus\{0\}}x^{-1}$. Since V_n is an $\mathbb{F}_q$-vector space, we have:

$$\prod_{x\in\ker(\phi_a)\setminus\{0\}}x=\left(\prod_{x\in V_n\setminus\{0\}}x\right)\left(\prod_{\gamma\in\mathbb{F}_q^\times}\prod_{x\in V_n}(\gamma x_0-x)\right)=-\left(\prod_{x\in V_n\setminus\{0\}}x\right)\left(\prod_{x\in V_n}(x_0-x)\right)^{q-1}.$$

By comparing the leading terms in the identity $(1-\tau)p_n=\alpha_n\phi_{a^n}$, we get that:

$$\begin{aligned}\alpha_n^{q-1}=&a^{-(q-1)n}\left(\prod_{x\in V_n}(x_0-x)\right)^{-q(q-1)}\left(\prod_{x\in\ker\phi_{a^n}\setminus\{0\}}x\right)^{q-1}\\=&-a^{-(q-1)n}\left(\prod_{x\in V_n\setminus\{0\}}x\right)^{q}\left(\prod_{x\in\ker\phi_{a^n}\setminus\{0\}}x\right)^{-1}\end{aligned}$$

For $i=1,\dots,rn$ choose recursively $y_i\in\ker\phi_{a^n}\setminus\mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j<i})$ as an element of maximum norm, and set $r_i:=\|y_i\|$; since x_0 is an element of maximum norm in $\ker\phi_{a^n}\setminus V_n$, and since $V_n\subseteq\ker\phi_{a^n}$ has codimension 1, we can assume without loss of generality that $y_k=x_0$ for some k and that $y_i\in V$ for all $i\neq k$; also, for $n\gg0$ we can assume that k does not depend on n. For all $i=1,\dots,rn$ the elements in $W_i:=\mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j\geq i})\setminus\mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j>i})$ all have norm r_i, and we have:

$$\#W_i=q^{rn+1-i}-q^{rn-i}\text{ and }\#(W_i\cap V_n)=\begin{cases}q^{rn-i}-q^{rn-1-i}&\text{if } i<k\\ q^{rn+1-i}-q^{rn-i}&\text{if } i>k\end{cases}.$$

We deduce the following identity for $n\gg0$:

$$\begin{aligned}\|\alpha_na^n\|^{q-1}=&\left\|\prod_{i=1}^{rn}\left(\left(\prod_{x\in W_i\cap V_n}x\right)^q\left(\prod_{x\in W_i}x\right)^{-1}\right)\right\|=\prod_{i=1}^{rn}\left(\left(\prod_{x\in W_i\cap V_n}r_i\right)^q\left(\prod_{x\in W_i}r_i\right)^{-1}\right)\\=&\left(\left(\prod_{i=1}^{k-1}r_i^{q^{rn-i}-q^{rn-1-i}}\right)\left(\prod_{i=k+1}^{rn}r_i^{q^{rn+1-i}-q^{rn-i}}\right)\right)^q\left(\prod_{i=1}^{rn}r_i^{q^{rn+1-i}-q^{rn-i}}\right)^{-1}\\=&r_{rn}^{1-q}\prod_{i=k}^{rn-1}\left(\frac{r_{i+1}}{r_i}\right)^{q^{rn+1-i}-q^{rn-i}}\leq r_{rn}^{1-q}\leq r_k^{1-q},\end{aligned}$$

which is constant; in particular, α_n tends to 0. For all $n\geq1$, we define $q_n=\sum_i(q_n)_i\tau^i\in\mathbb{C}_\infty[\tau]$ as the unique polynomial such that $p_{n+1}-p_n=q_n\phi_{a^n}$, which is well defined because $p_{n+1}-p_n|_{\ker\phi_{a^n}}=0$, and has degree in τ equal to $r-1$; we also set $q_0:=p_1$. For all $n\geq2$ we have:

$$(1-\tau)q_n\phi_{a^n}=(1-\tau)p_{n+1}-(1-\tau)p_n=\alpha_{n+1}\phi_{a^{n+1}}-\alpha_n\phi_{a^n}=(\alpha_{n+1}\phi_a-\alpha_n)\phi_{a^n},$$

hence $(1-\tau)q_n=\alpha_{n+1}\phi_a-\alpha_n$. Since α_n tends to 0, the maximum M_n of the norms of the coefficients on the right hand side tends to 0. We have $\|(q_n)_0\|\leq M_n$ and for all $i\geq0$ $\|(q_n)_{i+1}-(q_n)_i^q\|\leq M_n$, hence if $M_n<1$ we deduce that $\|(q_n)_i\|\leq M_n$ for all i.

We have deduced that $\sum_{n\geq 0}(q_n)_i \otimes a^n$ is a well defined element of $\mathbb{C}_\infty \hat{\otimes} A$ for all $i = 0, \dots, r-1$, hence we can define

$$\sum_{i=0}^{r-1}\left(\sum_{n\geq 0}(q_n)_i \otimes a^n\right)\cdot \tau^i \in \overline{M(E)}$$

with image $g \in \operatorname{Hom}^{cont}_{\mathbb{F}_q}(\exp_\phi(K_\infty\Lambda_\phi), \mathbb{C}_\infty)$. For any x in the domain, we have:

$$g(x) = \sum_i \sum_{n\geq 0}(q_n)_i \tau^i \phi_{a^n}(x) = \sum_{n\geq 0} q_n \circ \phi_{a^n}(x) = p_1(x) + \sum_{n\geq 1}(p_{n+1} - p_n)(x) = \lim_n p_n(x).$$

We claim that $g(x) = f$. On one hand, $g(x_0) = 1$; on the other hand, we need to prove that $g|_V = 0$. Since $\bigcup_{n\geq 1}\ker\phi_{a^n}$ is dense in $\exp_\phi(K_\infty\Lambda_\phi)$, it suffices to show that $g(x) = 0$ for all $x \in V \cap \bigcup_{n\geq 1}\ker\phi_{a^n} = \bigcup_{n\geq 1} V_n$. In this case, $x \in V_m$ for some $m \geq 1$, hence $p_n(x) = 0$ for all $n \geq m$ and $g(x) = \lim_n p_n(x) = 0$.

Since f was arbitrary, the image of the map $\overline{M(\underline{E})} \to \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$ contains the set of maps $\operatorname{Hom}^{cont}_{\mathbb{F}_q}(\exp_\phi(K_\infty\Lambda_\phi), \mathbb{F}_q) \cong \operatorname{Hom}_A(\Lambda_\phi, \Omega) \subseteq \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$, and since the map is $\mathbb{C}_\infty \hat{\otimes} A$-linear, it is surjective.

□

Remark 6.2.3. By the definition of the universal Anderson eigenvector $\omega_\phi \in \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$, the image of τ^i under the immersion $M(\underline{E}) \hookrightarrow \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$ is $(\tau^i \otimes 1)\omega_\phi = \omega_\phi^{(i)}$

Remark 6.2.4. For all $a \in A \setminus \mathbb{F}_q$, $(a \otimes 1 - 1 \otimes a)M(\underline{E}) \subseteq \tau M(\underline{E})$; in particular, since $(a \otimes 1 - 1 \otimes a)$ is invertible in $\mathbb{C}_\infty \hat{\otimes} A$, which is a flat extension of $A_{\mathbb{C}_\infty}$, the inclusion $\tau M(\underline{E}) \subseteq M(\underline{E})$ induces a natural isomorphism of left $A_{\mathbb{C}_\infty}[\tau]$-modules $\overline{\tau M(\underline{E})} \cong \overline{M(\underline{E})} \cong \mathbb{C}_\infty \hat{\otimes} \operatorname{Hom}_A(\Lambda_\phi, \Omega)$.

The dual A-motive $N(\underline{E})$ can be canonically identified with the right $A_{\mathbb{C}_\infty}[\tau]$-module $\mathbb{C}_\infty[\tau^{-1}]$, where $h \in \mathbb{C}_\infty[\tau]$ acts by composition on the left with h^*, and $a \in A$ acts by composition on the right with ϕ_a^*. Equivalently, $N(\underline{E})$ can be endowed with the structure of left $A_{\mathbb{C}_\infty}[\tau^{-1}]$-module.

Proposition 6.2.5. *Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld A-module. There is a natural immersion of right $A_{\mathbb{C}_\infty}[\tau]$-modules from $N(\underline{E})$ to $\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ which induces an isomorphism between $\overline{N(\underline{E})} := N(\underline{E}) \otimes_{A_{\mathbb{C}_\infty}} (\mathbb{C}_\infty \hat{\otimes} A)$ and $\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$.*

Proof. By Corollary 5.2.3 we have a natural map of right $A_{\mathbb{C}_\infty}[\tau]$-modules

$$\begin{aligned} N(\underline{E}) &= \operatorname{Hom}_{\mathbb{F}_q, \mathbb{C}_\infty}(\mathbb{G}_a, E(\mathbb{C}_\infty)) = \bigoplus_{k\geq 0} \operatorname{Hom}_k(\mathbb{C}_\infty, E(\mathbb{C}_\infty)) = \bigoplus_{k\geq 0} \operatorname{Hom}_{-k}(E(\mathbb{C}_\infty)^\vee, \mathbb{C}_\infty) \\ &\hookrightarrow \operatorname{Hom}^{cont}_{\mathbb{F}_q}(E(\mathbb{C}_\infty)^\vee, \mathbb{C}_\infty) \to \operatorname{Hom}^{cont}_{\mathbb{F}_q}(\ker(\exp_\phi^*), \mathbb{C}_\infty) \cong \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi, \end{aligned}$$

sending $f = \sum_i c_i \tau^{-i} \in N(\underline{E})$ to the restriction to $\ker(\exp_\phi^*) \subseteq E(\mathbb{C}_\infty)^\vee$ of the associated function $E(\mathbb{C}_\infty)^\vee \to \mathbb{C}_\infty$.

Let's prove that the induced map from $\overline{N(\underline{E})}$ to $\mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ is bijective, starting with injectivity; since $N(\underline{E})$ is a flat $A_{\mathbb{C}_\infty}$-module, this also proves that the map $N(\underline{E}) \hookrightarrow \overline{N(\underline{E})} \to \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ is an immersion.

First note that, for any $t \in A \setminus \mathbb{F}_q$, $\overline{N(\underline{E})} = N(\underline{E}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty \hat{\otimes} \mathbb{F}_q[t]$, hence any element $f \in \overline{N(\underline{E})}$ can be written in a unique way as $\sum_{i=0}^{r-1}\left(\sum_{n\geq 0} c_{i,n} \otimes t^n\right) \cdot \tau^{-i}$, where r is the degree of ϕ_t in τ.

Suppose that its image $f(\mathbb{C}_\infty) \in \mathrm{Hom}^{cont}_{\mathbb{F}_q}(\ker(\exp^*_\phi), \mathbb{C}_\infty)$ is identically zero. Fix some integer $m \geq 1$: for all $x \in \ker(\phi^*_{t^m}) \subseteq \ker(\exp^*_\phi)$ we have

$$0 = f(\mathbb{C}_\infty)(x) = \sum_{i=0}^{r-1} \left(\sum_{n=0}^{r-1} c_{i,n} \tau^{-i} \phi^*_{t^n} \right)(x),$$

and since $\#\ker(\phi^*_{t^m}) = q^{mr}$ and the q^{mr-1}-th power of the function on the right hand side is a polynomial of degree less than q^{mr}, it must be identically zero as an element of $\mathbb{C}_\infty[\tau^{-1}]$. Since the elements $\{\tau^{-i}\phi^*_{t^n}\} \subseteq \mathbb{C}_\infty[\tau^{-1}]$ all have different degrees, they are $\mathbb{C}_\infty$-linearly independent, hence $c_{i,n} = 0$ for all $i = 0, \dots, r-1$ and all $n = 0, \dots, m-1$. Finally, since m is arbitrary, we deduce that $f = 0$.

Let's prove surjectivity. Pick some $f : \ker(\exp^*_\phi) \to \mathbb{F}_q$ and call $V := \ker(f)$; fix some $a \in A$ such that $V \cap \ker \phi^*_a \subsetneq \ker \phi^*_a$ and define $V_n := V \cap \ker \phi^*_{a^n}$ for all $n \geq 1$: we know that $V_n \subsetneq \ker \phi^*_{a^n}$ has codimension 1. Denote by r the degree in τ of ϕ_a and pick some $x_0 \in \ker \phi^*_a$ of maximal norm such that $f(x_0) = 1$. For all $n \geq 1$, we define $p_n \in \mathbb{C}_\infty[\tau]$ as the unique additive polynomial of degree $nr - 1$ such that $p_n|_{V_n} = 0$ and $p_n(x_0) = 1$: in particular, $(1-\tau)p_n|_{\ker \phi^*_{a^n}} = 0$, so there is some $\alpha_n \in \mathbb{C}_\infty$ such that the polynomials $(1-\tau)p_n$ and $\tau^{rn}\alpha_n\phi^*_{a^n}$ coincide. Since $p_n(x_0) = 1$, the leading term of p_n is $\prod_{x \in V_n}(x_0 - x)^{-1}$. On the other hand, the leading term of $\tau^{rn}\phi^*_{a^n}$ is $a^{nq^{rn}}$. Since V_n is an $\mathbb{F}_q$-vector space, we have:

$$\prod_{x \in \ker(\phi^*_{a^n}) \setminus V_n} x = \left(\prod_{\gamma \in \mathbb{F}_q^\times} \prod_{x \in V_n} (\gamma x_0 - x) \right) = -\left(\prod_{x \in V_n} (x_0 - x) \right)^{q-1}.$$

By comparing the leading terms in the identity $(1-\tau)p_n = \tau^{rn}\alpha_n\phi^*_{a^n}$, we get that:

$$\alpha_n = a^{-n} \left(\prod_{x \in V_n} (x_0 - x) \right)^{-q^{1-rn}} = -a^{-n} \left(\prod_{x \in \ker(\phi^*_{a^n}) \setminus V_n} x \right)^{\frac{q^{1-rn}}{1-q}}$$

For $i = 1, \dots, rn$ choose recursively $y_i \in \ker \phi^*_{a^n} \setminus \mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j<i})$ as an element of maximum norm, and set $r_i := \|y_i\|$; since x_0 is an element of maximum norm in $\ker \phi^*_{a^n} \setminus V_n$, and since $V_n \subseteq \ker \phi_{a^n}$ has codimension 1, we can assume without loss of generality that $y_k = x_0$ for some k and that $y_i \in V$ for all $i \neq k$; also, for $n \gg 0$ we can assume that k does not depend on n. For all $i = 1, \dots, rn$ the elements in $W_i := \mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j \geq i}) \setminus \mathrm{Span}_{\mathbb{F}_q}(\{y_j\}_{j>i})$ all have norm r_i, and we have:

$$\#W_i = q^{rn+1-i} - q^{rn-i} \text{ and } \#(W_i \cap V_n) = \begin{cases} q^{rn-i} - q^{rn-1-i} & \text{if } i < k \\ q^{rn+1-i} - q^{rn-i} & \text{if } i > k \end{cases};$$

in particular, $W_i = W_i \cap V_n$ for $i > k$. We deduce the following identity for $n \gg 0$:

$$\begin{aligned}
\|\alpha_n a^n\|^{(q-1)q^{nr-1}} &= \left\| \prod_{i=1}^{rn} \left(\left(\prod_{x \in W_i \cap V_n} x \right) \left(\prod_{x \in W_i} x \right)^{-1} \right) \right\| = \left\| \prod_{i=1}^{k} \left(\left(\prod_{x \in W_i \cap V_n} x \right) \left(\prod_{x \in W_i} x \right)^{-1} \right) \right\| \\
&= \prod_{i=1}^{k} \left(\left(\prod_{x \in W_i \cap V_n} r_i \right) \left(\prod_{x \in W_i} r_i \right)^{-1} \right) = r_k^{-q^{rn-k}(q-1)} \prod_{i=1}^{k-1} r_i^{-q^{rn-1-i}(q-1)^2} \\
\Rightarrow \|\alpha_n a^n\| &= r_k^{-q^{1-k}} \prod_{i=1}^{k-1} r_i^{-q^{-i}(q-1)},
\end{aligned}$$

which is constant; in particular, α_n tends to 0. For all $n \geq 1$, we define $q_n = \sum_i (q_n)_i \tau^i \in \mathbb{C}_\infty[\tau]$ as the unique polynomial such that $p_{n+1} - \tau^r p_n = q_n \tau^{rn} \phi^*_{a^n}$, which is well defined because $p_{n+1} - \tau^r p_n|_{\ker \phi^*_{a^n}} = 0$, and has degree in τ equal to $r-1$; we also set $q_0 := p_1$. For all $n \geq 2$ we have:

$$\begin{aligned}(1-\tau) q_n \tau^{rn} \phi^*_{a^n} &= (1-\tau) p_{n+1} - (1-\tau)\tau^r p_n \\ &= \tau^{r(n+1)} \alpha_{n+1} \phi^*_{a^{n+1}} - \tau^r \tau^{rn} \alpha_n \phi^*_{a^n} \\ &= \tau^{r(n+1)} (\alpha_{n+1} \phi^*_a - \alpha_n) \phi^*_{a^n},\end{aligned}$$

hence $(1-\tau) q_n \tau^{rn} = \tau^{r(n+1)}(\alpha_{n+1}\phi^*_a - \alpha_n)$. Since α_n tends to 0, the maximum M_n of the norms of the coefficients on the right hand side tends to 0. We have $\|(q_n)_0\| \leq M_n$ and for all $i \geq 0$ $\|(q_n)_{i+1} - (q_n)_i^q\| \leq M_n$, hence if $M_n < 1$ we deduce that $\|(q_n)_i\| \leq M_n$ for all i.

We have deduced that $\sum_{n \geq 0} (q_n)_i^{q^{1-r}} \otimes a^n$ is a well defined element of $\mathbb{C}_\infty \hat{\otimes} A$ for all $i = 0, \ldots, r-1$, hence we can define

$$\sum_{i=0}^{r-1} \left(\sum_{n \geq 0} (q_n)_i^{q^{1-r}} \otimes a^n \right) \cdot \tau^{1+i-r} \in \overline{N(\underline{E})}$$

with image $g \in \mathrm{Hom}^{cont}_{\mathbb{F}_q}(\ker(\exp^*_\phi), \mathbb{C}_\infty)$. For any x in the domain, we have:

$$\begin{aligned} g(x) &= \sum_{i=0}^{r-1} \sum_{n\geq 0} (q_n)_i^{q^{1-r(n+1)}} \tau^{1+i-r} \phi^*_{a^n}(x) = \sum_{n \geq 0} \tau^{1-r(n+1)} q_n \tau^{rn} \circ \phi^*_{a^n}(x) \\ &= \tau \left(\tau^r p_1(x) + \sum_{n \geq 1} (\tau^{-r(n+1)} p_{n+1} - \tau^{rn} p_n)(x) \right) = \lim_n (\tau^{1-rn} p_n)(x) \end{aligned}$$

We claim that $g(x) = f$. On one hand, $g(x_0) = 1$; on the other hand, we need to prove that $g|_V = 0$. Since $\bigcup_{n \geq 1} \ker \phi^*_{a^n}$ is dense in $\ker(\exp^*_\phi)$, it suffices to show that $g(x) = 0$ for all $x \in V \cap \bigcup_{n \geq 1} \ker \phi_{a^n} = \bigcup_{n \geq 1} V_n$. In this case, $x \in V_m$ for some $m \geq 1$, hence $p_n(x) = 0$ for all $n \geq m$ and $g(x) = \lim_n (\tau^{1-rn} p_n)(x) = 0$.

Since f was arbitrary, the image of the map $\overline{N(\underline{E})} \to \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ contains the set of continuous $\mathbb{F}_q$-linear maps $\mathrm{Hom}^{cont}_{\mathbb{F}_q}(\ker(\exp^*_\phi), \mathbb{F}_q) \cong \Lambda_\phi \subseteq \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$, and since the map is $\mathbb{C}_\infty \hat{\otimes} A$-linear, it is surjective. ☐

Remark 6.2.6. By the definition of the universal dual Anderson eigenvector $\zeta_\phi \in \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$, the image of τ^{-i} under the immersion $N(\underline{E}) \hookrightarrow \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$ is $(\tau^{-i} \otimes 1)\zeta_\phi = \zeta_\phi^{(-i)}$

Theorem 6.2.7. *Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld A-module. Under the natural immersions of $A_{\mathbb{C}_\infty}$-modules $\tau M(\underline{E}) \subseteq \mathbb{C}_\infty \hat{\otimes} \mathrm{Hom}_A(\Lambda_\phi, \Omega)$ and $N(\underline{E}) \subseteq \mathbb{C}_\infty \hat{\otimes} \Lambda_\phi$, the $\mathbb{C}_\infty$-linear map $\Theta : N(\underline{E}) \to \tilde{N}(\underline{E})$ sending ζ to the map $\omega \mapsto \zeta \cdot \omega$ is an isomorphism of dual A-motives.*

Proof. We simply need to show that Θ is an isomorphism of right $A_{\mathbb{C}_\infty}[\tau]$-modules. Let's first prove that Θ is well defined: we can write any $\zeta \in N(\underline{E})$ and any $\omega \in \tau M(\underline{E})$ as finite sums $\sum_{j \geq 0} c_j \zeta_\phi^{(-j)}$ and $\sum_{i \geq 0} d_i \omega_\phi^{(i+1)}$ respectively, hence $\Theta(\zeta)(\omega) = \sum_{j,i \geq 0} c_j d_i \zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}$, which belongs to $\Omega_{\mathbb{C}_\infty}$ by Theorem 5.4.2.

By Lemma 5.4.1, the dot product is $A_{\mathbb{C}_\infty}$-linear, hence so is Θ. Let's check that Θ commutes with the right action of τ. Recall that for all $j \geq 0$ $\zeta_\phi^{(-j)} \cdot \tau = \zeta_\phi^{(-j-1)}$ and that for all $f \in \tilde{N}(\underline{E}) =$

$\mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau\mathbb{C}_\infty[\tau], \Omega_{\mathbb{C}_\infty})$ we have $(f\cdot\tau)(m) = (f(\tau\cdot m))^{(-1)}$. For all $j, i \geq 0$ we have:

$$\begin{aligned}\Theta(\zeta_\phi^{(-j)}\cdot\tau)(\omega_\phi^{(i+1)}) = \Theta(\zeta_\phi^{(-j-1)})(\omega_\phi^{(i+1)}) = \zeta_\phi^{(-j-1)}\cdot\omega_\phi^{(i+1)} = \left(\zeta_\phi^{(-j)}\cdot\omega_\phi^{(i+2)}\right)^{(-1)}\\ = \left(\Theta(\zeta_\phi^{(-j)})(\tau\omega_\phi^{(i+1)})\right)^{(-1)} = \left(\Theta(\zeta_\phi^{(-j)})\cdot\tau\right)(\omega_\phi^{(i+1)}).\end{aligned}$$

If we tensor $\Theta : N(\underline{E}) \to \mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \Omega_{\mathbb{C}_\infty})$ by $\mathbb{C}_\infty\hat{\otimes}A$ we get the morphism

$$\begin{aligned}\tilde{\Theta} : \mathbb{C}_\infty\hat{\otimes}\Lambda_\phi \cong \overline{N(\underline{E})} \to \overline{\mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \Omega_{\mathbb{C}_\infty})} = \mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \mathbb{C}_\infty\hat{\otimes}\Omega)\\ = \mathrm{Hom}_{\mathbb{C}_\infty\hat{\otimes}A}(\overline{\tau M(\underline{E})}, \mathbb{C}_\infty\hat{\otimes}\Omega) \cong \mathrm{Hom}_{\mathbb{C}_\infty\hat{\otimes}A}(\mathbb{C}_\infty\hat{\otimes}\,\mathrm{Hom}_A(\Lambda_\phi, \Omega), \mathbb{C}_\infty\hat{\otimes}\Omega),\end{aligned}$$

sending an element ζ to the map sending $\omega \in \mathbb{C}_\infty\hat{\otimes}\,\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ to $\zeta\cdot\omega$; in particular, $\tilde{\Theta}$ is an isomorphism. Since the maps $N(\underline{E}) \hookrightarrow \overline{N(\underline{E})}$ and $\mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \Omega_{\mathbb{C}_\infty}) \hookrightarrow \overline{\mathrm{Hom}_{A_{\mathbb{C}_\infty}}(\tau M(\underline{E}), \Omega_{\mathbb{C}_\infty})}$ are injective, and $\tilde{\Theta}$ is an isomorphism, we deduce that Θ is injective.

We claim that the action of τ on $\mathrm{coker}(\Theta)$ is invertible. Let's fix an $A_{\mathbb{C}_\infty}$-linear function $f : \tau M(\underline{E}) \to \Omega_{\mathbb{C}_\infty}$ such that $f\cdot\tau = \Theta(\zeta)$ for some $\zeta = \sum_{j\geq 0} c_j\zeta_\phi^{(-j)} \in N(\underline{E})$: we want to prove that $c_0 = 0$, so that f is also in the image of Θ. Fix any $a \in A\setminus\mathbb{F}_q$, and write $\phi_a - a = \sum_{k\geq 1} a_k\tau^k$. We have:

$$\begin{aligned}(1\otimes a - a^q\otimes 1)\omega_\phi^{(1)} =& \tau\left((1\otimes a)\omega_\phi - (a\otimes 1)\omega_\phi\right) = \tau\left(\sum_{k\geq 1}(a_k\otimes 1)\omega_\phi^{(k)}\right) = \sum_{k\geq 1}(a_k^q\otimes 1)\omega_\phi^{(k+1)}\\ \Rightarrow f(\omega_\phi^{(1)}) =& (1\otimes a - a^q\otimes 1)^{-1} f\left(\sum_{k\geq 1}(a_k^q\otimes 1)\omega_\phi^{(k+1)}\right) =\\ =& (1\otimes a - a^q\otimes 1)^{-1}\left((f\cdot\tau)\left(\sum_{k\geq 1}(a_k\otimes 1)\omega_\phi^{(k)}\right)\right)^{(1)}\\ =& (1\otimes a - a^q\otimes 1)^{-1}\left(\sum_{k\geq 1}(a_k\otimes 1)\zeta\cdot\omega_\phi^{(k)}\right)^{(1)}\\ =& \zeta^{(1)}\cdot\left((1\otimes a - a^q\otimes 1)^{-1}\left(\sum_{k\geq 1}(a_k^q\otimes 1)\omega_\phi^{(k+1)}\right)\right)\\ =& (\zeta\cdot\omega_\phi)^{(1)} = \left(\sum_{j\geq 0} c_j\zeta_\phi^{(-j)}\cdot\omega_\phi\right)^{(1)}.\end{aligned}$$

Since $\zeta_\phi^{(-j)}\cdot\omega_\phi \in \Omega_{\mathbb{C}_\infty}$ for all $j > 0$, and since $\zeta_\phi\cdot\omega_\phi \notin \Omega_{\mathbb{C}_\infty}$, we have $c_0 = 0$.

Let's denote by $\mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta))$ the set of associated primes of $\mathrm{coker}(\Theta)$, i.e. the set of prime ideals $\mathfrak{p} < A_{\mathbb{C}_\infty}$ such that there is $x \in \mathrm{coker}(\Theta)$ with annihilator $\mathrm{Ann}_{A_{\mathbb{C}_\infty}}(x) = \mathfrak{p}$. Since $\tau : C \to C$ is bijective, for all $x \in C$ $\mathrm{Ann}_{A_{\mathbb{C}_\infty}}(x\cdot\tau) = \mathrm{Ann}_{A_{\mathbb{C}_\infty}}(x)^{(1)}$, hence $\mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta))$ is closed under Frobenius twist; on the other hand, since $A_{\mathbb{C}_\infty}$ is noetherian and $\mathrm{coker}(\Theta)$ is a finitely generated $A_{\mathbb{C}_\infty}$-module, $\mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta))$ is finite. We deduce that there is some positive integer k such that for all $\mathfrak{p} \in \mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta))$ $\mathfrak{p}^{(k)} = \mathfrak{p}$, i.e. $\mathfrak{p}$ is the extension of some ideal in $A_{\mathbb{F}_{q^k}}$. For any such prime $\mathfrak{p}$, the associated map $A_{\mathbb{F}_{q^k}} \to A_{\mathbb{F}_{q^k}}/\mathfrak{p}\cap A_{\mathbb{F}_{q^k}}$ can be extended continuously to $\mathbb{C}_\infty\hat{\otimes}A$, hence the extended ideal $(\mathbb{C}_\infty\hat{\otimes}A)\mathfrak{p}$ is proper. In particular, since $\mathbb{C}_\infty\hat{\otimes}A$ is a flat extension of $A_{\mathbb{C}_\infty}$, $\mathrm{Ass}_{\mathbb{C}_\infty\hat{\otimes}A}(\overline{\mathrm{coker}(\Theta)}) \cong$

$\mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta))$; since $\overline{\mathrm{coker}(\Theta)} \cong \mathrm{coker}(\tilde{\Theta}) = 0$, we deduce $\mathrm{Ass}_{A_{\mathbb{C}_\infty}}(\mathrm{coker}(\Theta)) = 0$, hence $\mathrm{coker}(\Theta)$ has no torsion. On the other hand, since $N(\underline{E})$ and $\tilde{N}(\underline{E})$ are both projective $A_{\mathbb{C}_\infty}$-modules of the same rank, and since Θ is injective, $\mathrm{coker}(\Theta)$ is a torsion $A_{\mathbb{C}_\infty}$-module, which means it is 0. □

Remark 6.2.8. Since $N(\underline{E})$ is a right $\mathbb{C}_\infty[\tau]$-module of rank 1, $\Theta : N(\underline{E}) \to \tilde{N}(\underline{E})$ coincides with the isomorphism described by Hartl and Juschka up to a scalar factor in $\mathbb{C}_\infty$.

In particular, the computation of the Hartl-Juschka pairing HJ becomes the same problem as the computation of the dot products $\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}$, which is tackled in Section 5.4.

In the case of a Drinfeld $\mathbb{F}_q[t]$-module we can check by direct computation that the morphism of dual t-motives Θ defined in Theorem 6.2.7 is the same as the one defined by Hartl and Juschka.

Let's start with the explicit computation by Hartl and Juschka.

Proposition 6.2.9 ([HJ20][Ex. 2.5.16])**.** *Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld $\mathbb{F}_q[t]$-module of rank r, with $\phi_t = \sum_i t_i \tau^i$. Let $\{\alpha_{i,j}\}_{0\le i,j<r} \in \mathbb{C}_\infty^{r\times r}$ be the matrix with entries $\alpha_{i,j} := -t_{i+j+1}^{q^{-i}}$, which is invertible, and let $\{\beta_{i,j}\}_{0\le i,j<r} \in \mathbb{C}_\infty^{r\times r}$ be its inverse. Then for all $0 \le i,j < r$, the following identity holds:*

$$HJ(\tau^{-j} \otimes \tau^{i+1}) = \beta_{i,j} dt.$$

We can now prove the following theorem.

Theorem 6.2.10. *Let $\underline{E} = (\mathbb{G}_a, \phi)$ be a Drinfeld $\mathbb{F}_q[t]$-module. The following identity holds in $\Omega_{\mathbb{C}_\infty}$ for all $i,j \ge 0$:*

$$HJ(\tau^{-j} \otimes \tau^{i+1}) = \zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}.$$

Proof. By Remarks 6.2.3 and 6.2.6, we can identify $\tau M(\underline{E}) = \mathrm{Span}_{\mathbb{C}_\infty}\{\omega_\phi^{(i+1)}\}_{i\ge 0}$ and $N(\underline{E}) = \mathrm{Span}_{\mathbb{C}_\infty}\{\zeta_\phi^{(-j)}\}_{j\ge 0}$, hence we need to prove that the dot product coincides with the Hartl–Juschka pairing.

If we call r the rank of ϕ, the set $\{\tau^{i+1}\}_{0\le i<r}$ generates $\tau M(\underline{E})$ as an $A_{\mathbb{C}_\infty}$-module, and the set $\{\tau^{(-j)}\}_{0\le j<r}$ generates $N(\underline{E})$ as an $A_{\mathbb{C}_\infty}$-module; since both the Hartl–Juschka pairing and the dot product are $A_{\mathbb{C}_\infty}$-linear, it suffices to prove the statement for all $0 \le i,j < r$.

By Proposition 6.2.9, we need the following identity to hold in $\Omega_{\mathbb{C}_\infty} \subseteq \mathbb{C}_\infty \hat{\otimes} \Omega$ for all $0 \le i,j < r$:

$$\sum_{i=0}^{r-1} (t_{k+i+1} \otimes 1)^{q^{-k}} \left(\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}\right) = -\delta_{k,j} dt.$$

If $k > j$, we have:

$$\begin{aligned}\sum_{i=0}^{r-1} (t_{k+i+1} \otimes 1)^{q^{-k}} \left(\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}\right) &= \sum_{i=r-k}^{r-1} (t_{k+i+1} \otimes 1)^{q^{-k}} \left(\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}\right) \\ &+ \sum_{i=0}^{r-1-k} (t_{k+i+1} \otimes 1)^{q^{-k}} \left(\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)}\right) = 0,\end{aligned}$$

where the first sum is 0 because $t_l = 0$ if $l > r$, and the second sum is 0 because, by Proposition 5.5.1, $\zeta_\phi^{(-j)} \cdot \omega_\phi^{(i+1)} = \left(\zeta_\phi \cdot \omega_\phi^{(i+j+1)}\right)^{(-j)} = 0$ if $0 < i+j+1 < r$, which is true because $i,j \ge 0$ and $i+1 \le r-k < r-j$.

Since ω_ϕ is a Anderson eigenvector, the identity $\sum_l (t_l \otimes 1)\omega_\phi^{(l)} = \omega_\phi(1 \otimes t)$ holds, hence if $k \leq j$ we have:

$$\left(\sum_{i=0}^{r-1}(t_{k+i+1}\otimes 1)^{q^{-k}}\left(\zeta_\phi^{(-j)}\cdot\omega_\phi^{(i+1)}\right)\right)^{(k)} = \zeta_\phi^{(k-j)}\cdot\sum_{i=k+1}^{r+k}(t_i\otimes 1)\omega_\phi^{(i)} = \zeta_\phi^{(k-j)}\cdot\sum_{i=k+1}^{r}(t_i\otimes 1)\omega_\phi^{(i)}$$

$$=(1\otimes t - t\otimes 1)\zeta_\phi^{(k-j)}\cdot\omega_\phi - \zeta_\phi^{(k-j)}\cdot\sum_{i=1}^{k}(t_i\otimes 1)\omega_\phi^{(i)}$$

$$=(1\otimes t - t\otimes 1)\left(\zeta_\phi\cdot\omega_\phi^{(j-k)}\right)^{(k-j)} - \sum_{i=1}^{k}(t_i\otimes 1)\left(\zeta_\phi\cdot\omega_\phi^{(i+j-k)}\right)^{(k-j)}.$$

By Proposition 5.5.1, since $0 < i+j-k \leq j < r$, the sum on the right hand side is 0, while $(1\otimes t - t\otimes 1)\left(\zeta_\phi\cdot\omega_\phi^{(j-k)}\right)^{(k-j)}$ is 0 if $k<j$ and $-dt$ if $k=j$. $\square$

6.2.2 The case of Anderson A-modules

Finally, in Question 6.2.11 we point to a possible generalization of Theorem 6.2.7 in the hypothesis of a uniformizable abelian (and A-finite) Anderson A-module $\underline{E}$. Since $\underline{E}$ is uniformizable, we have the following natural isomorphism of $A_{\mathbb{C}_\infty}$-modules (cf. [HJ20][Def. 2.4.14, Prop. 2.4.17, Thm. 2.5.32]):

$$\tau M(\underline{E})\otimes_{A_{\mathbb{C}_\infty}}(\mathbb{C}_\infty\hat{\otimes}A) = M(\underline{E})\otimes_{A_{\mathbb{C}_\infty}}(\mathbb{C}_\infty\hat{\otimes}A) = \mathbb{C}_\infty\hat{\otimes}\Lambda_\phi. \tag{6.1}$$

In [HJ20][Def. 2.3.18], the right hand side isomorphism of 6.1 is actually given as the definition of uniformizability for the A-motive $M(\underline{E})$; the left hand side isomorphism holds because the elements $\{a\otimes 1 - 1\otimes a\}_{a\in A\setminus\mathbb{F}_q}\subseteq A_{\mathbb{C}_\infty}$ are invertible in $\mathbb{C}_\infty\hat{\otimes}A$. Similarly, we have the following natural isomorphism of $A_{\mathbb{C}_\infty}$-modules (cf. [HJ20][Def. 2.4.14, Prop. 2.4.17, Thm. 2.5.32]):

$$N(\underline{E})\otimes_{A_{\mathbb{C}_\infty}}(\mathbb{C}_\infty\hat{\otimes}A) = \mathbb{C}_\infty\hat{\otimes}\operatorname{Hom}_A(\Lambda_\phi,\Omega). \tag{6.2}$$

Due to the importance that the spaces $E(\mathbb{C}_\infty)\hat{\otimes}\operatorname{Hom}_A(\Lambda_\phi,\Omega)$ and $E(\mathbb{C}_\infty)^\vee\hat{\otimes}\Lambda_\phi$ have in the context of (dual) Anderson eigenvectors, it's reasonable to tensor the isomorphisms 6.1 and 6.2 by $E(\mathbb{C}_\infty)$ and $E(\mathbb{C}_\infty)^\vee$, respectively, obtaining:

$$E(\mathbb{C}_\infty)\hat{\otimes}\operatorname{Hom}_A(\Lambda_\phi,\Omega) = \tau M(\underline{E})\otimes_{A_{\mathbb{C}_\infty}}(E(\mathbb{C}_\infty)\hat{\otimes}A); \tag{6.3}$$

$$E(\mathbb{C}_\infty)^\vee\hat{\otimes}\Lambda_\phi = N(\underline{E})\otimes_{A_{\mathbb{C}_\infty}}(E(\mathbb{C}_\infty)^\vee\hat{\otimes}A). \tag{6.4}$$

On one hand the two $A_{\mathbb{C}_\infty}$-modules on the left hand side of the isomorphisms 6.3 and 6.4 can be paired by a natural map, denoted by $\boxtimes$, which is a generalization of the dot product introduced in Lemma 5.4.1:

$$\boxtimes : \left((E(\mathbb{C}_\infty))\hat{\otimes}\operatorname{Hom}_A(\Lambda_\phi,\Omega)\right)\otimes_{A_{\mathbb{C}_\infty}}\left(E(\mathbb{C}_\infty)^\vee\hat{\otimes}\Lambda_\phi\right)\to\operatorname{End}_{\mathbb{C}_\infty}(E(\mathbb{C}_\infty))\hat{\otimes}\Omega.$$

On the other hand, looking at the objects on the right hand side of the isomorphisms 6.3 and 6.4, $\tau M(\underline{E})$ and $N(\underline{E})$ can be paired via the Hartl-Jushka map HJ, and we can also define the following natural $A_{\mathbb{C}_\infty}$-linear map:

$$\Gamma : \left(E(\mathbb{C}_\infty)\hat{\otimes}A\right)\otimes_{A_{\mathbb{C}_\infty}}\left(E(\mathbb{C}_\infty)^\vee\hat{\otimes}A\right)\to\operatorname{End}_{\mathbb{C}_\infty}(E(\mathbb{C}_\infty))\hat{\otimes}A.$$

All of this allows us to ask the following question.

Question 6.2.11. Does the pairing $\boxtimes$ coincide with $HJ \otimes \Gamma$?

Remark 6.2.12. If we assume $\underline{E} = (\mathbb{G}_a, \phi)$ to be a Drinfeld module, $E(\mathbb{C}_\infty) = E(\mathbb{C}_\infty)^\vee = \mathbb{C}_\infty$, and the pairing map Γ is simply the multiplication of $\mathbb{C}_\infty \hat{\otimes} A$. Since $E(\mathbb{C}_\infty)\hat{\otimes} A \cong E(\mathbb{C}_\infty)^\vee \hat{\otimes} A \cong \mathbb{C}_\infty \hat{\otimes} A$, the isomorphisms 6.3 and 6.4 coincide with the ones proven in Proposition 6.2.2 and Proposition 6.2.5. Under these isomorphisms, the restriction of the pairing $\boxtimes$, which is the dot product, does coincide with the Hartl–Juschka pairing—up to a scalar factor in $\mathbb{C}_\infty$—by Theorem 6.2.7.

6.3 Conjectures for Anderson A-modules

Both the Anderson eigenvectors (Definition 2.2.7) and the dual Anderson eigenvectors (Definition 5.2.8) are well defined in the context of abelian Anderson A-modules. Due to the asymmetry between the hypotheses of Theorem 2.2.9 and Theorem 5.2.10, it's natural to ask the following question.

Question 6.3.1. Let $\underline{E} = (E, \phi)$ be a uniformizable abelian (and A-finite) Anderson A-module. Is the functor Sf_{ϕ^*} represented by Λ_ϕ?

Throughout the rest of this section, we work with a uniformizable abelian Anderson A-module $\underline{E} = (E, \phi)$ and we assume that Question 6.3.1 has an affirmative answer. Under this assumption, we denote by $\zeta_\phi \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$ the universal dual Anderson eigenvector, i.e. the universal object of Sf_{ϕ^*}. As usual, we also denote by ω_ϕ the universal Anderson eigenvector.

Our aim is to formulate a possible generalization of Theorem 5.4.17 to abelian (and A-finite) Anderson A-modules (Question 6.3.2); while this conjectural result is at the moment not well-defined (see Remark 6.3.3), we use it as a stepping stone to formulate some smaller reasonable conjectures and prove some propositions which generalize many intermediate results used in the proof of Theorem 5.4.17.

6.3.1 The main question

For any pair of elements $\omega \in E(\mathbb{C}_\infty)\hat{\otimes}\,\mathrm{Hom}_A(\Lambda_\phi, \Omega)$ and $\zeta \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$, let's denote by $\omega \circ \zeta$ the pairing $\omega \boxtimes \zeta$; the reason for this notation is that, if we write $\omega = \sum_i v_i \otimes h_i$ and $\zeta = \sum_j w_j^* \otimes \lambda_j$, $\omega \boxtimes \zeta = \sum_{i,j} (v_i \circ w_j^*) \otimes h_i(\lambda_j)$.

Recall the notation established in Definition 5.2.1. For any element

$$f = (f_k)_k \in \prod_{k \in \mathbb{Z}} \mathrm{End}_k(\mathbb{C}_\infty) = \mathbb{C}_\infty[[\tau, \tau^{-1}]],$$

we use the following notation:

$$\begin{aligned} \omega \circ (f \otimes 1) \circ \zeta \coloneqq \left(\sum_{i,j} (v_i \circ f_k \circ w_j^*) \otimes h_i(\lambda_j) \right)_k &\in \prod_{k\in\mathbb{Z}} \left(\mathrm{End}_k(E(\mathbb{C}_\infty)) \hat{\otimes} \Omega \right) \\ &\cong \prod_{k \in \mathbb{Z}} \mathrm{Hom}^{cont}_{\mathbb{F}_q} \left(K_\infty/A, \mathrm{End}_k(E(\mathbb{C}_\infty)) \right), \end{aligned}$$

which is well defined. For any x in this space and for any $c \in K_\infty$, we denote by $x(c)$ the image of c in $\prod_{k\in\mathbb{Z}} \mathrm{End}_k(E(\mathbb{C}_\infty))$.

For any $\mathbb{C}_\infty$-vector space V, let's denote by $\overline{\mathrm{End}}(V)$ the ring $\bigoplus_{j\in\mathbb{Z}} \prod_{k>j} \mathrm{End}_k(V)$ (when $V = \mathbb{C}_\infty^d$, $\overline{\mathrm{End}}(V) = \mathbb{C}_\infty^{d\times d}[[\tau]][\tau^{-1}]$).

Question 6.3.2. Let $\Phi : K_\infty \to \overline{\mathrm{End}}(E(\mathbb{C}_\infty))$ and $\hat{\Phi} : K_\infty \to \overline{\mathrm{End}}(E(\mathbb{C}_\infty)^\vee)$ be ring homomorphisms which extend ϕ and ϕ^* respectively, and are continuous on each coordinate, and define $\mathcal{T} := (\tau^k)_k \in \prod_{k\in\mathbb{Z}} \mathrm{End}_k(\mathbb{C}_\infty)$. Consider the following identities in $\prod_{k\in\mathbb{Z}} \mathrm{End}_k(E(\mathbb{C}_\infty))$ for all $c \in K$:

$$\left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi\right)(c) = \Phi_c - (\hat{\Phi}_c)^*.$$

Do they hold?

Remark 6.3.3. This question is actually ill-posed in many cases when $\underline{E}$ is not a Drinfeld module. The reason is that the leading term of ϕ_a^* may be non-invertible for some $a \in A$, hence it may be impossible to extend ϕ^* to K multiplicatively. For example, when $A = \mathbb{F}_q[t]$ and $\underline{E} = (\mathbb{G}_a^d, C^{\otimes d})$ is the d-th tensor power of the Carlitz module (see Definition 6.4.1), we have:

$$(C_t^{\otimes d})^* = \begin{pmatrix} t & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & t \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \tau^{-1},$$

and the second matrix, having rank 1, is not invertible if $d > 1$.

On the other hand, Φ is actually well defined. We can write $\exp_\phi : \mathrm{Lie}(E) \to E(\mathbb{C}_\infty)$ as an element $(E_k)_k$ of the infinite product

$$\prod_{k\geq 0} \mathrm{Hom}_k(\mathrm{Lie}(E), E(\mathbb{C}_\infty));$$

since $E_0 \in \mathrm{Hom}_{\mathbb{C}_\infty}(\mathrm{Lie}(E), E(\mathbb{C}_\infty))$ is invertible, being the natural isomorphism of the two $\mathbb{C}_\infty$-vector spaces, we can define the logarithm

$$\log_\phi = (L_k)_k \in \prod_{k\geq 0} \mathrm{Hom}_k(E(\mathbb{C}_\infty), \mathrm{Lie}(E))$$

as the inverse of $\exp_\phi$.

In particular, just like in the case of Drinfeld modules, the unique ring homomorphism $\Phi : K_\infty \to \overline{\mathrm{End}}(E(\mathbb{C}_\infty))$ which extends ϕ and is continuous on each coordinate is the one sending $c \in K_\infty$ to $\exp_\phi \circ c \circ \log_\phi$; in other words, for all $k \in \mathbb{Z}$:

$$(\Phi_c)_k = \sum_i E_i \circ c \circ L_{k-i}.$$

6.3.2 Conjectures and propositions

Despite Remark 6.3.3, it's useful to use Question 6.3.2 as a guide to better understand the nature of the object $(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi) \in \prod_{k\in\mathbb{Z}} \mathrm{Hom}^{cont}_{\mathbb{F}_q}\left({K_\infty}/{A}, \mathrm{End}_k(E(\mathbb{C}_\infty))\right)$.

For example, following the proof of Theorem 5.4.17, we are able to prove the following property.

Proposition 6.3.4. *The following identities hold in $\prod_{k\in\mathbb{Z}} \mathrm{End}_k(E(\mathbb{C}_\infty))$ for all $a \in A$ and for all $c \in K$:*

$$\left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi\right)(ac) = \phi_a \circ \left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi\right)(c) = \left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi\right)(c) \circ \phi_a.$$

Proof. We have the following chain of identities in $\prod_{k\in\mathbb{Z}} \mathrm{End}_k(E(\mathbb{C}_\infty))$ for all $a \in A$ and for all $c \in K$:

$$\begin{aligned}
(\phi_a \otimes 1) \circ (\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi)(c) &= \left(\left(\sum_i ((\phi_a)_i \otimes 1)\omega_\phi\right) \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi\right)(c) \\
&= ((\omega_\phi(1 \otimes a)) \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi)(c) \\
&=(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi)(ac) \\
&=(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ (\zeta_\phi(1 \otimes a)))(c) \\
&= \left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \left(\sum_i ((\phi_a)_i^* \otimes 1)\zeta_\phi\right)\right)(c) \\
&= \left(\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \left(\sum_i (\tau^{-i} \otimes 1) \circ \zeta_\phi \circ ((\phi_a)_i \otimes 1)\right)\right)(c) \\
&= \left(\sum_i \omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi \circ ((\phi_a)_i \otimes 1)\right)(c) \\
&= (\omega_\phi \circ (\mathcal{T} \otimes 1) \circ \zeta_\phi)(c) \circ (\phi_a \otimes 1).
\end{aligned}$$

□

Recall the identity $(\Phi_c)_k = \sum_i E_i \circ c \circ L_{k-i}$ shown in Remark 6.3.3. Another property used in the proof of Theorem 5.4.17 is Proposition 5.1.19. We formulate the following conjecture, which generalizes it.

Conjecture 6.3.5. *For any positive real number R there is $k_0 \in \mathbb{Z}$ such that for all $k \geq k_0$, for all $c \in K_\infty$ with $\|c\| \leq R$:*

$$\sum_i E_i \circ c \circ L_{k-i} = \left(\omega_\phi \circ (\tau^k \otimes 1) \circ \zeta_\phi\right)(c).$$

Remark 6.3.6. In the proof of Proposition 5.1.19, an important step is Lemma 4.3.21, which expresses the coefficients of the logarithm as series of negative powers of the elements of the lattice Λ_ϕ. Those identities make use of the factorization property for entire functions in $\mathbb{C}_\infty[[x]]$; the factorization of entire functions in the ring $(\mathbb{C}_\infty[[x_1, \ldots, x_d]])^d$ is not as well understood. Moreover, for an Anderson A-module of dimension $d > 1$, negative powers of elements of the period lattice are not well-defined, so it's not possible to formulate a naive generalization of Lemma 4.3.21.

By the previous remark, we need to give a different interpretation to Lemma 4.3.21. A useful point of view is to think of it as establishing a relation between the universal Anderson eigenvector ζ_ϕ and the coefficients of the logarithm $\log_\phi$ when $\underline{E} = (\mathbb{G}_a, \phi)$, as expressed for example in Proposition 5.3.1. In the context of Anderson A-modules, we can formulate a more general conjecture.

First, let's give a Definition which is analogous to the property for an element of $\mathbb{C}_\infty \hat{\otimes} A$ to be an entire function.

Definition 6.3.7. Let's consider a series in $\sum_i v_i \otimes \lambda_i \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$, where $\{\lambda_i\}_i$ is an $\mathbb{F}_q$-linear basis of Λ_ϕ. We say it is *quickly converging* if, for given a norm $|\cdot|$ on $E(\mathbb{C}_\infty)^\vee$ and a norm $\|\cdot\|$ on $\mathrm{Lie}(E) \supseteq \Lambda_\phi$, for any integer k, $\lim_i |v_i|^{q^k} \cdot \|\lambda_i\| = 0$.

Remark 6.3.8. Note that all norms on finite $\mathbb{C}_\infty$-vector spaces are equivalent, so the notion of quick convergence is independent from the choice of norm on $E(\mathbb{C}_\infty)^\vee$ and $\mathrm{Lie}(E)$. By Proposition 5.3.1, if $E = (\mathbb{G}_a, \phi)$, ζ_ϕ is quickly convergent.

Conjecture 6.3.9. *The universal dual Anderson eigenvector $\zeta_\phi \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$ is quickly converging. Moreover, if we write $\zeta_\phi = \sum_i z_i \otimes \lambda_i \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$, where $\{\lambda_i\}_i$ is an $\mathbb{F}_q$-linear basis of Λ_ϕ, for all $k \in \mathbb{Z}$ we have $\sum_i \lambda_i \circ \tau^k \circ z_i = (\log_\phi)_k$.*

Assuming the convergence property, we are able to prove, similarly to the proof of Proposition 5.3.1, that the element $\left(\sum_i \lambda_i \circ \tau^k \circ z_i\right)_k$ in $\prod_{k\in\mathbb{Z}} \mathrm{Hom}_k(E(\mathbb{C}_\infty), \mathrm{Lie}(E))$ satisfies the same functional identity as the logarithm. In other words, the following proposition holds.

Proposition 6.3.10. *Let $\zeta = \sum_i z_i \otimes \lambda_i \in E(\mathbb{C}_\infty)^\vee \hat{\otimes} \Lambda_\phi$ be a dual Anderson eigenvector, where $\{\lambda_i\}_i$ is an $\mathbb{F}_q$-linear basis of Λ_ϕ, and assume it is quickly converging. The following identity holds in $\prod_{k\in\mathbb{Z}} \mathrm{Hom}_k(E(\mathbb{C}_\infty), \mathrm{Lie}(E))$ for all $a \in A$:*

$$\mathrm{Lie}(\phi_a) \circ \left(\sum_{i,k} \lambda_i \circ \tau^k \circ z_i \right) = \left(\sum_{i,k} \lambda_i \circ \tau^k \circ z_i \right) \circ \phi_a.$$

Proof. We have the following chain of identities for all $a \in A$:

$$\mathrm{Lie}(\phi_a) \circ \sum_{i,k} \lambda_i \circ \tau^k \circ z_i = \sum_k \sum_i (a\lambda_i) \circ \tau^k \circ z_i = \sum_k \sum_{i,j} \lambda_i \circ \tau^k \circ z_i \circ (\phi_a)_j = \sum_{i,k} \lambda_i \circ \tau^k \circ z_i \circ \phi_a,$$

where the second identity holds by Remark 5.2.9. □

Remark 6.3.11. Assuming that Conjecture 6.3.9 holds, it's possible to give a partial proof of Conjecture 6.3.5.

If we write $\omega_\phi = \sum_j v_j \otimes h_j$, where $(h_j)_{j\geq 0}$ is an $\mathbb{F}_q$-basis of $\mathrm{Hom}_{\mathbb{F}_q}(\Lambda_\phi, \Omega) = \widehat{K_\infty\Lambda_\phi/\Lambda_\phi}$, we can write:

$$\left(\omega_\phi \circ (\tau^k \otimes 1) \circ \zeta_\phi\right)(c) = \sum_{i,j} h_j(c \cdot \lambda_i) v_j \circ \tau^k \circ z_i,$$

which is well defined because the sequences $(v_j)_j$ and $(z_i)_i$ converge to 0 in $E(\mathbb{C}_\infty)$ and $E(\mathbb{C}_\infty)^\vee$, respectively. Since ω_ϕ is the exponential, as a function from $K_\infty\Lambda_\phi/\Lambda_\phi$ to $E(\mathbb{C}_\infty)$, for any element $x \in K_\infty\Lambda_\phi$ we have $\exp_\phi(x) = \sum_j h_j(x)v_j$. On the other hand, as an element of $E(\mathbb{C}_\infty) = \mathrm{Hom}_{\mathbb{C}_\infty}(\mathbb{C}_\infty, E(\mathbb{C}_\infty))$, we can write $\exp_\phi(x) = \sum_j E_j \circ x \circ \tau^{-j}$, hence:

$$(\omega_\phi \circ (\tau^k \otimes 1) \circ \zeta_\phi)(c) = \sum_{i,j} h_j(c \cdot \lambda_i) v_j \circ \tau^k \circ z_i = \sum_i \sum_j E_j \circ c \circ \lambda_i \circ \tau^{k-j} \circ z_i.$$

Since by Conjecture 6.3.9 $\sum_j \sum_i E_j \circ c \circ \lambda_i \circ \tau^{k-j} \circ z_i = \sum_j E_j \circ c \circ L_{k-j}$, to prove Conjecture 6.3.5 it would suffice to show that for any given $c \in K_\infty$, for $k \gg 0$, it's possible to swap the two infinite sums.

6.4 The case of the Carlitz tensor power

Let's start by defining the tensor power of an arbitrary Anderson A-module.

Definition 6.4.1. Let $\underline{E} = (E, \phi)$ be an Anderson A-module. The d-th tensor power $\underline{E'} = (E', \phi^{\otimes d})$ of (E, ϕ) is defined as the unique Anderson A-module such that $M(\underline{E'}) = M(\underline{E})^{\otimes d}$, where the tensor product on the right is $A_{\mathbb{C}_\infty}$-linear.

Remark 6.4.2. The period lattice $\Lambda_{\phi^{\otimes d}}$ is naturally isomorphic to $\Lambda_\phi^{\otimes d}$ as an A-module (see for example [HJ20][Prop. 2.3.24.c, Thm. 2.5.32]), hence the tensor power of $\mathbb{C}_\infty\hat{\otimes}A$-modules $(\mathbb{C}_\infty\hat{\otimes}\Lambda_\phi)^{\otimes d}$ is naturally isomorphic to $\mathbb{C}_\infty\hat{\otimes}\Lambda_{\phi^{\otimes d}}$.

The d-th tensor power of the Carlitz module is the simplest example of an Anderson module of dimension $d > 1$, and has been thoroughly studied since the seminal paper of Anderson and Thakur [AT90]. For this reason it's a natural case study to test the conjectures of the previous section.

From now on we assume $A = \mathbb{F}_q[t]$, and denote the d-th tensor power of the Carlitz module by $(\mathbb{G}_a, C^{\otimes d})$. We have the following identities in $\mathrm{End}_{\mathbb{C}_\infty}(\mathbb{C}_\infty^d)[\tau]$:

$$C_t^{\otimes d} = \begin{pmatrix} t & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdot & 0 & t \end{pmatrix} + \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}\tau, \ \ \mathrm{Lie}\, C_t^{\otimes d} = \begin{pmatrix} t & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdot & 0 & t \end{pmatrix}.$$

For simplicity, and to align ourselves with the notation of previous articles on the subject, we denote by t the variable $1 \otimes t \in \mathbb{C}_\infty\hat{\otimes}A$ and by θ the variable $t \otimes 1 \in \mathbb{C}_\infty\hat{\otimes}A$.

Proposition 6.4.3. *The following element is a dual Anderson eigenvector for the Anderson A-module $(\mathbb{G}_a, C^{\otimes d})$:*

$$\zeta := ((t-\theta)^{d-j}\zeta_C^{\otimes d})_{j=1,\dots,d} \in \mathbb{C}_\infty^d\hat{\otimes}\Lambda_{C^{\otimes d}}.$$

If the functor $\mathrm{Sf}_{C^{\otimes d*}}$ is represented by $\Lambda_{C^{\otimes d}}$, its universal object is equal to ζ up to a factor in $\mathbb{F}_q^\times$.

Proof. Under the isomorphism $\Lambda_C \cong A$, we can identify $\mathbb{C}_\infty\hat{\otimes}\Lambda_C$ and $\mathbb{C}_\infty\hat{\otimes}\Lambda_{C^{\otimes d}}$ with the ring $\mathbb{C}_\infty\hat{\otimes}A$, so we can identify $\zeta_C^{\otimes d}$ with ζ_C^d. We can write:

$$(C_\theta^{\otimes d})^* = +\begin{pmatrix} \theta & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \theta \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}\tau^{-1}.$$

For $j = 1, \dots, d$, we have:

$$\begin{aligned} ((C_\theta^{\otimes d})^*(\zeta))_j &= \begin{cases} \theta\zeta_1 + \zeta_d^{(-1)} \text{ if } j = 1 \\ \theta\zeta_j + \zeta_{j-1} \text{ if } j > 1 \end{cases} \\ &= \begin{cases} \theta(t-\theta)^{d-1}\zeta_C^d + (\zeta_C^d)^{(-1)} = \theta(t-\theta)^{d-1}\zeta_C^d + (t-\theta)^d\zeta_C^d \text{ if } j = 1 \\ \theta(t-\theta)^{d-j}\zeta_C^d + (t-\theta)^{d-j+1}\zeta_C^d \text{ if } j > 1 \end{cases} \\ &= t(t-\theta)^{d-j}\zeta_C^d = t\zeta_j, \end{aligned}$$

hence ζ is an Anderson eigenvector.

If $\mathrm{Sf}_{C^{\otimes d*}}$ is represented by $\Lambda_{C^{\otimes d}}$ with universal object $\zeta_{C^{\otimes d}}$, since $\Lambda_{C^{\otimes d}} \cong \Lambda_C \cong A$, we have $\zeta = a\zeta_{C^{\otimes d}}$ for some $a \in A$. Since $\zeta_C \in \mathbb{C}_\infty\hat{\otimes}A$ is invertible, for all $a \in A \setminus \mathbb{F}_q$ the product $(1 \otimes a^{-1})\zeta_C^d$ does not belong to $\mathbb{C}_\infty\hat{\otimes}A$, hence ζ is equal to $\zeta_{C^{\otimes d}}$ up to a factor in $\mathbb{F}_q^\times$. $\square$

If we assume that Question 6.3.1 has an affirmative answer, $\zeta := ((t-\theta)^{d-j}\zeta_C^{\otimes d})_{j=1,\dots,d}$ is the universal Anderson eigenvector for the Anderson A-module $\underline{E} := (\mathbb{G}_a, C^{\otimes d})$, and the next proposition proves Conjecture 6.3.9 for $\underline{E}$.

Let's first include a Lemma due to Papanikolas. For all integers $k \geq 0$, denote by $\partial_t^{(k)} : \mathbb{C}_\infty\hat{\otimes}A \to \mathbb{C}_\infty\hat{\otimes}A$ the k-th Hasse derivative in the variable t, and denote by L_k the k-th coefficient of the logarithm $\log_{C^{\otimes d}} \in \mathrm{End}_{\mathbb{C}_\infty}(\mathbb{C}_\infty^d)[[\tau]] = \prod_{k\geq 0}\mathrm{End}_k(\mathbb{C}_\infty^d)$ associated to the d-th tensor power of the Carlitz module.

Lemma 6.4.4 ([Pap15]). *For all $k \geq 0$, for $1 \leq i,j \leq d$, the following identity holds in $\mathbb{C}_\infty$:*

$$(L_k)_{i,j} = (-1)^d\left[\partial_t^{(d-j)}\left((t-\theta^{q^k})^{d-i}\left((t-\theta^q)\cdots(t-\theta^{q^k})\right)^{-d}\right)\right]_{t=\theta}.$$

Proposition 6.4.5. *Fix a generator $\bar{\lambda}$ of the A-module $\Lambda_{C^{\otimes d}}$. For $i = 1,\dots,d$, write:*

$$(t-\theta)^{d-i}\zeta_C^{\otimes d} = \sum_{n\geq 0}((t-\theta)^{d-i}\zeta_C^{\otimes d})_{(n)} \otimes (t^n\cdot\bar{\lambda})_i, \text{ where } ((t-\theta)^{d-i}\zeta_C^{\otimes d})_{(n)} \in \mathbb{C}_\infty^d \forall n.$$

For all $k \in \mathbb{Z}$, for $1 \leq i,j \leq d$, the following identity holds in $\mathbb{C}_\infty$:

$$(L_k)_{i,j} = \sum_n (t^n\cdot\bar{\lambda})_j((t-\theta)^{d-i}\zeta_C^{\otimes d})_{(n)}^{q^k}.$$

Proof. For $k = 1,\dots,d$, we can express the k-th coordinate of $\bar{\lambda} \in \mathbb{C}_\infty^d$ as follows (see for example [Mau22][Eq. 3]):

$$\bar{\lambda}_k = \left[(-1)^d\partial_t^{(d-k)}(\zeta_C^{-d})\right]_{t=\theta}.$$

Starting from the explicit expression of the matrix $\mathrm{Lie}\, C_\theta^{\otimes d}$, we can derive the following identities for the entries of the matrix $\mathrm{Lie}\, C_a^{\otimes d}$ for all $a \in A$, for $1 \leq i,j \leq d$:

$$(\mathrm{Lie}\, C_a^{\otimes d})_{i,j} = \begin{cases}\partial_\theta^{(j-i)}(a) \text{ if } i \leq j\\ 0 \text{ if } i > j\end{cases}.$$

We deduce the following identity for the k-th coordinate of $t^n\cdot\bar{\lambda}$, for $k = 1,\dots,d$:

$$(t^n\cdot\bar{\lambda})_k = \left(\mathrm{Lie}\, C_{\theta^n}^{\otimes d}(\bar{\lambda})\right)_k = \sum_{h=0}^{d-k}\partial_\theta^{(h)}(\theta^n)\lambda_{k+h}.$$

Using the Leibniz rule for the Hasse derivative and the identity $\zeta_C^{(-1)} = (t-\theta)\zeta_C$, we get the following chain of identities in $\mathbb{C}_\infty$ for all $k \geq 0$, for $1 \leq i, j \leq d$:

$$
\begin{aligned}
\sum_n (t^n \cdot \bar{\lambda})_j ((t-\theta)^{d-i} \zeta_C^{\otimes d})^{q^k}_{(n)} =& \sum_n \left(\left((t-\theta)^{d-i}\zeta_C^d\right)^{q^k}_{(n)} \sum_{h=0}^{d-j} \partial_\theta^{(h)}(\theta^n)\bar{\lambda}_{j+h} \right) \\
=& \sum_n \left(\left((t-\theta)^{d-i}\zeta_C^d\right)^{q^k}_{(n)} \sum_{h=0}^{d-j} \theta^{n-h}\binom{n}{h}\bar{\lambda}_{j+h} \right) \\
=& \sum_{h=0}^{d-j} \sum_n \left(\binom{n}{h} \left((t-\theta)^{d-i}\zeta_C^d\right)^{q^k}_{(n)} \theta^{n-h} \right) \bar{\lambda}_{j+h} \\
=& \sum_{h=0}^{d-j} \left[\partial_t^{(h)} \left((t-\theta)^{d-i}\zeta_C^d\right)^{(k)} \right]_{t=\theta} \left[(-1)^d \partial_t^{(d-j-h)}(\zeta_C^{-d}) \right]_{t=\theta} \\
=& (-1)^d \left[\partial_t^{(d-j)} \left((t-\theta^{q^k})^{d-i} \left(\zeta_C^{(k)}\zeta_C^{-1}\right)^d \right) \right]_{t=\theta} \\
=& (-1)^d \left[\partial_t^{(d-j)} \left((t-\theta^{q^k})^{d-i} \left((t-\theta)\zeta_C^{(k)}\omega_C\right)^d \right) \right]_{t=\theta} \\
=& (-1)^d \left[\partial_t^{(d-j)} \left((t-\theta^{q^k})^{d-i} \left((t-\theta^q)\cdots(t-\theta^{q^k})\right)^{-d} \right) \right]_{t=\theta},
\end{aligned}
$$

which coincides with the formula for the coefficients of the logarithm as expressed in Lemma 6.4.4. When $k < 0$, for $1 \leq i, j \leq d$, all the identities hold except the last one, and we get:

$$
\begin{aligned}
\sum_n (t^n \cdot \bar{\lambda})_j ((t-\theta)^{d-i} \zeta_C^{\otimes d})^{q^k}_{(n)} =& (-1)^d \left[\partial_t^{(d-j)} \left((t-\theta^{q^k})^{d-i} \left((t-\theta)\zeta_C^{(k)}\omega_C\right)^d \right) \right]_{t=\theta} \\
=& (-1)^d \left[\partial_t^{(d-j)} \left((t-\theta^{q^k})^{d-i} \left((t-\theta^{q^{k+1}})\cdots(t-\theta)\right)^d \right) \right]_{t=\theta},
\end{aligned}
$$

which is zero because the $(d-j)$-th hyperderivative of a multiple of $(t-\theta)^d$ is a multiple of $(t-\theta)^j$. $\square$

Bibliography

[And86] Greg W. Anderson. “t-motives”. In: *Duke Mathematical Journal* 53.2 (1986), pp. 457–502. DOI: 10.1215/S0012-7094-86-05328-7.

[And94] Greg W. Anderson. “Rank one elliptic A-modules and A-harmonic series”. In: *Duke Mathematical Journal* 73.3 (1994), pp. 491–542. DOI: 10.1215/S0012-7094-94-07321-3.

[And96] Greg W. Anderson. “Log-algebraicity of twisted A-harmonic series and special values of L-series in characteristic p”. In: *Journal of Number Theory* 60.1 (1996), pp. 165–209. DOI: 10.1006/jnth.1996.0119.

[ABP04] Greg W. Anderson, W. Dave Brownawell, and Mattew A. Papanikolas. “Determination of the algebraic relations among special Γ-values in positive characteristic”. In: *Annals of Mathematics* 160.1 (2004), pp. 237–313. DOI: 10.4007/annals.2004.160.237.

[AT90] Greg W. Anderson and Dinesh S. Thakur. “Tensor powers of the Carlitz module and zeta values”. In: *Annals of Mathematics* 132.1 (1990), pp. 159–191. DOI: 10.2307/1971503.

[ANT17a] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. “Special functions and twisted L-series”. In: *Journal de Théorie des Nombres de Bordeaux* 29.3 (2017), pp. 931–961. DOI: 10.5802/jtnb.1007.

[ANT17b] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. “Stark units in positive characteristic”. In: *Proceedings of the London Mathematical Society* 115.4 (2017), pp. 763–812. DOI: 10.1112/plms.12051.

[ANT22] Bruno Anglès, Tuan Ngo Dac, and Floric Tavares Ribeiro. “A class formula for admissible Anderson modules”. In: *Inventiones mathematicae* 229 (2022), pp. 1–44. DOI: 10.1007/s00222-022-01110-3.

[AP14] Bruno Anglès and Federico Pellarin. “Universal Gauss-Thakur sums and L-series”. In: *Inventiones mathematicae* 200.2 (2014), pp. 653–669. DOI: 10.1007/s00222-014-0546-8.

[Art24] Emil Artin. “Quadratische Körper im Gebiete der höheren Kongruenzen I, II.” In: *Mathematische Zeitschrift* 19 (1924), pp. 153–206, 207–246.

[Car35] Leonard Carlitz. “On certain functions connected with polynomials in a Galois field”. In: *Duke Mathematical Journal* 1.2 (1935), pp. 137–168. DOI: 10.1215/S0012-7094-35-00114-4. URL: https://doi.org/10.1215/S0012-7094-35-00114-4.

[Car37] Leonard Carlitz. “An analogue of the von Staudt–Clausen theorem”. In: *Duke Mathematical Journal* 3.3 (1937), pp. 503–517. DOI: 10.1215/S0012-7094-37-00340-5.

[Car40] Leonard Carlitz. "An analogue of the Staudt–Clausen theorem". In: *Duke Mathematical Journal* 7.1 (1940), pp. 62–67. DOI: 10.1215/S0012-7094-40-00703-7. URL: https://doi.org/10.1215/S0012-7094-40-00703-7.

[CNP23] Kwun Chung, Tuan Ngo Dac, and Federico Pellarin. "Universal families of Eulerian multiple zeta values in positive characteristic". In: *Advances in Mathematics* 422 (2023). DOI: 10.1016/j.aim.2023.109003.

[Dri74] Vladimir G. Drinfeld. "Elliptic modules". In: *Matematicheskii Sbornik (Novaya Seriya)* 94(136).4(8) (1974), pp. 594–627. DOI: 10.1070/SM1974v023n04ABEH001731. English version: "Elliptic modules". In: *Mathematics of the USSR-Sbornik* 23.4 (1974), pp. 561–592. DOI: 10.1215/S0012-7094-40-00703-7.

[GM21] Quentin Gazda and Andreas Maurischat. "Special functions and Gauss–Thakur sums in higher rank and dimension". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2021.773 (2021), pp. 231–261. DOI: 10.1515/crelle-2020-0038.

[Gek88] Ernst-Ulrich Gekeler. "On the coefficients of Drinfeld modular forms". In: *Inventiones mathematicae* 93.3 (1988), pp. 667–700. DOI: 10.1007/BF01410204.

[GP19] Oğuz Gezmiş and Matthew A. Papanikolas. "The de Rham isomorphism for Drinfeld modules over Tate algebras". In: *Journal of Algebra* 525 (2019), pp. 454–496. DOI: 10.1016/j.jalgebra.2019.02.006.

[Gos78] David Goss. "Von Staudt for $\mathbf{F}_q[T]$". In: *Duke Mathematical Journal* 45.4 (1978), pp. 885–910. DOI: 10.1215/S0012-7094-78-04541-6.

[Gos79] David Goss. "v-adic Zeta Functions, L-series and Measures for Function Fields." In: *Inventiones mathematicae* 55 (1979), pp. 107–116. DOI: 10.1007/BF01390084.

[Gos92] David Goss. "L-series of t-motives and Drinfeld Modules". In: *The Arithmetic of Function Fields*. Ed. by David Goss, David R. Hayes, and Michael I. Rosen. Berlin, New York: De Gruyter, 1992, pp. 313–402. ISBN: 9783110886153. DOI: 10.1515/9783110886153.313.

[Gos98] David Goss. *Basic Structures of Function Field Arithmetic*. Vol. 35. Ergebnisse der Mathematik und ihrer Grenzgebeite. Berlin, Heidelberg: Springer, 1998. DOI: 10.1007/978-3-642-61480-4.

[GP18] Nathan Green and Matthew A. Papanikolas. "Special L-values and shtuka functions for Drinfeld modules on elliptic curves". In: *Research in the Mathematical Sciences* 5.1 (2018). DOI: 10.1007/s40687-018-0122-8.

[HJ20] Urs Hartl and Ann-Kristin Juschka. "Pink's theory of Hodge structures and the Hodge conjecture over function fields". In: *t-Motives: Hodge Structures, Transcendence and Other Motivic Aspects*. EMS Series of Congress Reports. 2020, pp. 31–182. DOI: 10.4171/198-1/2.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, NY, 1977. ISBN: 9780387902449.

[Has36] Helmut Hasse. "Zur Theorie der abstrakten elliptischen Funktionenkörper I, II, III." In: *Journal für die reine und angewandte Mathematik* 175 (1936), pp. 55–62, 69–81, 193–207.

[Hay74] David R. Hayes. "Explicit class field theory for rational function fields". In: *Transactions of the American Mathematical Society* 189 (1974), pp. 77–91. DOI: 10.1090/S0002-9947-1974-0330106-6.

[Hay79] David R. Hayes. "Explicit class field theory in global function fields". In: *Studies in algebra and number theory*. Vol. 6. Adv. Math. Suppl. Stud. Academic Press, New York-London, 1979, pp. 173–217. ISBN: 0-12-599153-3.

[KP23] Chalinee Khaochim and Matthew A. Papanikolas. "Effective rigid analytic trivializations for Drinfeld modules". In: *Canadian Journal of Mathematics* 75.3 (2023), pp. 713–742. DOI: 10.4153/S0008414X22000153.

[Mau22] Andreas Maurischat. "Algebraic independence of the Carlitz period and its hyperderivatives". In: *Journal of Number Theory* 240.1 (2022), pp. 145–162. DOI: 10.1016/j.jnt.2022.01.006.

[Mau24] Andreas Maurischat. "Abelian equals A-finite for Anderson A-modules". Preprint available at: https://arxiv.org/abs/2110.11114. 2024.

[Mil86] James S. Milne. "Jacobian varieties". In: *Arithmetic Geometry*. Ed. by Gary Cornell and Joseph H. Silverman. New York, NY: Springer New York, 1986, pp. 167–212. DOI: 10.1007/978-1-4613-8655-1_7.

[Pap15] Matthew A. Papanikolas. "Log-algebraicity on tensor powers of the Carlitz module and special values of Goss L-functions". Preprint, 167 pages. 2015.

[Pel08] Federico Pellarin. "Aspects de l'indépendance algébrique en caractéristique non nulle [d'après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu,...]" In: *Séminaire Bourbaki - Volume 2006/2007 - Exposés 967-981*. Vol. 317. Astérisque. Paris: Société mathématique de France, 2008, pp. 205–242. DOI: 10.24033/ast.754.

[Pel12] Federico Pellarin. "Values of certain L-series in positive characteristic". In: *Annals of Mathematics* 176.3 (2012), pp. 2055–2093. DOI: 10.4007/annals.2012.176.3.13.

[Pon34] Lev S. Pontryagin. "The Theory of Topological Commutative Groups". In: *Annals of Mathematics* 35.2 (1934), pp. 361–388. DOI: 10.2307/1968438.

[Poon96] Bjorn Poonen. "Fractional power series and pairings on Drinfeld modules". In: *Journal of the American Mathematical Society* 9.3 (1996), pp. 783–812. DOI: 10.1090/S0894-0347-96-00203-2.

[Rie59] Bernhard Riemann. "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse". In: *Monatsberichte der Berliner Akademie* (1859), pp. 671–680.

[Tae12] Lenny Taelman. "Special L-values of Drinfeld modules". In: *Annals of Mathematics* 175.1 (2012), pp. 369–391. DOI: 10.4007/annals.2012.175.1.10.

[Tha88] Dinesh S. Thakur. "Gauss sums for $F_q[T]$". In: *Inventiones mathematicae* 94.1 (1988), pp. 105–112. DOI: 10.1007/BF01394346.

[Tha93] Dinesh S. Thakur. "Shtukas and Jacobi sums". In: *Inventiones mathematicae* 111.1 (1993), pp. 557–570. DOI: 10.1007/BF01231298.

[Wad41] Luther I. Wade. "Certain quantities transcendental over $GF(p^n, x)$". In: *Duke Mathematical Journal* 8.4 (1941), pp. 701–720. DOI: 10.1215/S0012-7094-41-00860-8. URL: https://doi.org/10.1215/S0012-7094-41-00860-8.

[Wei48] André Weil. *Sur les courbes algébriques et les variétés qui s'en deduisent*. Vol. 1048. Actualités scientifiques et industrielles. Hermann, Paris, 1948.