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The Direct Summand Theorem

Master Thesis
in Mathematics

Candidate
Giacomo Hermes Ferraro

Advisor
Tamas Szamuely
Università di Pisa

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Contents

Introduction

In this thesis, our aim is to present a recent result proved by Yves André in [And18], building on some results from [Bha14]: Hochster's direct summand conjecture in commutative algebra; we will focus on the approach given by Bhargav Bhatt in his article [Bha18], in which he streamlines André's original proof.

Hochster's original conjecture - now André's theorem - is the following:

Theorem 1 (André)**.** *Let A_0 be a regular ring, and let $A_0 \hookrightarrow B_0$ be a module-finite extension of A_0. This inclusion splits as a map of A_0-modules.*

Many partial solutions to Hochster's conjecture have been known for quite a long time. For example, Hochster himself proved that we can assume A_0 to be local and we may replace it by a faithfully flat extension in [Hoc73]. In that same article, Hochster solved the conjecture in the case of equal characteristic, i.e. when A_0 contains a copy of a field. The first chapter will focus on these simplifications, and will contain a brief proof of the two equal characteristic cases. In their articles, André and Bhatt proved the remaining case of characteristic $(0, p)$, which will be the focus of the rest of the this thesis.

The first approach to simplify the problem is as follows. The short exact sequence of A_0-modules $0 \to A_0 \to B_0 \to Q_0 \to 0$ corresponds to an element $\alpha_0 \in \mathrm{Ext}^1_{A_0}(A_0, Q_0)$, and the sequence splits if and only if $\alpha_0 = 0$. If we consider a faithfully flat extension $A_0 \to A$ and apply the tensor product we obtain a new short exact sequence $0 \to A \to B \to Q \to 0$, with corresponding element $\alpha \in \mathrm{Ext}^1_A(A, Q)$. The induced map $\mathrm{Ext}^1_{A_0}(A_0, Q_0) \to \mathrm{Ext}^1_A(A, Q)$ sends α_0 to α, and by faithful flatness $\alpha = 0$ if and only if $\alpha_0 = 0$, so it is sufficient to prove that the tensored sequence splits. In particular, this happens if $A \hookrightarrow B$ is étale, and since we know that there is some element $g \in A_0$ (the discriminant) such that $A_0[g^{-1}] \hookrightarrow B_0[g^{-1}]$ is étale, we would hope to find a suitable faithfully flat extension A such that from the étaleness of $A[g^{-1}] \hookrightarrow B[g^{-1}]$ we may deduce the étaleness of $A \hookrightarrow B$. It turns out that this requirement would be too strong, but there is a similar result given by Faltings' almost purity theorem, in the context of almost mathematics.

Almost mathematics allows - after the extension of A_0 to a suitably large ring A - to consider the more flexible category of almost-A-modules. This theory, first developed by Gerd Faltings in [Fal88], will be explored in Chapter 2, with the transposition of many definitions and properties in this new language (starting with almost zero modules and concluding with almost finite étale extensions) - the reference book for this argument is a comprehensive work by Gabber and Ramero: [GR03]. This is the preliminary work needed to state Faltings' almost-purity theorem, which for a suitable A states that if the map $A \to B$ is étale after inverting p, it is almost étale.

To apply this theorem, we will need to find some further extensions A_m where g divides p^m, so that inverting p inverts g, too. To solve this problem, and to give a nice formulation of Faltings' almost-purity theorem, we will introduce in Chapter 3 the concepts of perfectoid fields and algebras, first developed by Scholze in [Sch12]. Unfortunately, the extensions A_m won't

be faithfully flat over A: we will need a perfectoid version of Riemann's theorem on removable singularities - Scholze's *Hebbarkeitssatz*, which is proved in Chapter 4 - to retrieve étaleness back from these rings.

Finally, Chapter 5 will be devoted to the proof of the direct summand theorem, by presenting together all the previous constructions.

Chapter 1

Hochster's original work

In this chapter we will follow Hochster's first steps towards the solution of his conjecture, which naturally lead to the solution of the case of equal characteristic.

1.1 Strengthening of the hypothesis

First of all, let's state the direct summand theorem, as it was originally conjectured by Hochster.

Theorem 1.1.1 (André)**.** *Let A be a regular ring, and let $A \hookrightarrow B$ be a module-finite extension of A. This inclusion splits as a map of A-modules.*

The first and foremost result from Hochster is the following, which establishes that his conjecture is a local problem.

Proposition 1.1.2. *Let $A \hookrightarrow B$ be a module-finite extension of noetherian rings. Then:*

1. *the inclusion $A \hookrightarrow B$ splits (as a map of A-modules) if and only if $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ splits (as a map of $A_{\mathfrak{p}}$-modules) for every $\mathfrak{p} \subseteq A$ prime ideal;*

2. *similarly, if $A \hookrightarrow A'$ is a faithfully flat extension, $A \hookrightarrow B$ splits if and only if $A' \hookrightarrow B \otimes_A A'$ splits.*

Before proceeding, let's remind what we mean by *faithfully flat* extension with the following definition and lemma.

Definition 1.1.3. Let R be a commutative ring, M an R-module. The module M is faithfully flat if for every sequence $N_1 \to N_2 \to N_3$, it is exact at N_2 if and only if $M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3$ is exact at $M \otimes N_2$. An extension $R \to R'$ is faithfully flat if R' is faithfully flat as an R-module.

Lemma 1.1.4. *Let R be a commutative ring, M a flat R-module. The following are equivalent:*

1. *the module M is faithfully flat;*

2. *for every pair of R-modules P and Q the natural map $\mathrm{Hom}_R(P, Q) \to \mathrm{Hom}_R(M \otimes P, M \otimes Q)$ is injective;*

3. *for every R-module N, $M \otimes N = 0$ if and only if $N = 0$.*

Proof. First, let's prove (2) $\Leftrightarrow$ (3). Assume (2) is true, and take an R-module N such that $N \otimes M = 0$. We have an injective map:

$$N \cong \mathrm{Hom}_R(R, N) \hookrightarrow \mathrm{Hom}_R(R \otimes M, N \otimes M) \cong \mathrm{Hom}_R(M, 0) \cong 0,$$

therefore $N = 0$.

Assume (3) is true. To prove (2), we need to show that for any two R-modules P, Q and any map $f : P \to Q$, if the induced map $f \otimes id_M : P \otimes M \to Q \otimes M$ is zero, then $f = 0$. We can write $f = g \circ h$, with $h : P \to \mathrm{im}(f)$ surjective and $g : \mathrm{im}(f) \to Q$ injective. Since M is flat, $g \otimes id_M$ is injective and $h \otimes id_M$ is surjective, so $f \otimes id_M = 0$ if and only if $\mathrm{im}(f) \otimes M = 0$, which by hypothesis implies $\mathrm{im}(f) = 0$, i.e. $f = 0$.

Now let's prove (1) $\Leftrightarrow$ (3). Assume (3) is true and take $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$, a sequence of R-modules such that $M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3$ is exact. Since M is flat, we have that:

$$\frac{\ker(g)}{\mathrm{im}(f)} \otimes M \cong \frac{\ker(g) \otimes M}{\mathrm{im}(f) \otimes M} \cong \frac{\ker(g \otimes M)}{\mathrm{im}(f \otimes M)} \cong 0,$$

therefore $\frac{\ker(g)}{\mathrm{im}(f)} = 0$ and $N_1 \to N_2 \to N_3$ is exact.

Vice versa, assume (1) is true and consider the sequence $0 \to N \to 0$. If $M \otimes N = 0$, the tensored sequence is $0 \to 0 \to 0$, which is exact: this implies that $0 \to N \to 0$ is exact, so $N = 0$. □

We need the following lemma.

Lemma 1.1.5. *Let R be a commutative ring and M a finitely presented R-module. If R' is a flat ring over R, then for every R-module N the following natural map is an isomorphism:*

$$\alpha_{M,N} : \mathrm{Hom}_R(M, N) \otimes_R R' \to \mathrm{Hom}_R(M, N \otimes_R R').$$

Proof. If $M \cong R^n$ for some finite n, both arguments are naturally isomorphic to $(N \otimes_R R')^n$ and the isomorphisms carry $\alpha_{M,N}$ to the identity map.

In general, since M is finitely presented, there is an exact sequence $R^k \to R^n \to M \to 0$. The functors $\mathrm{Hom}_R(\cdot, N) \otimes_R R'$ and $\mathrm{Hom}_R(\cdot, N \otimes_R R')$ are both left exact, thus we get the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M, N) \otimes_R R' & \longrightarrow & \mathrm{Hom}_R(R^n, N) \otimes_R R' & \longrightarrow & \mathrm{Hom}_R(R^k, N) \otimes_R R' \\
\| & & \downarrow{\scriptstyle \alpha_{M,N}} & & \downarrow{\scriptstyle \alpha_{R^n,N}} & & \downarrow{\scriptstyle \alpha_{R^k,N}} \\
0 & \longrightarrow & \mathrm{Hom}_R(M, N \otimes_R R') & \longrightarrow & \mathrm{Hom}_R(R^n, N \otimes_R R') & \longrightarrow & \mathrm{Hom}_R(R^k, N \otimes_R R').
\end{array}$$

The diagram commutes because α is a natural transformation, and since $\alpha_{R^n,N}$ and $\alpha_{R^k,N}$ are both isomorphism, $\alpha_{M,N}$ is also an isomorphism by the five lemma. □

Corollary 1.1.6. *Using the properties of the tensor product, we can also deduce a natural isomorphism:*

$$\mathrm{Hom}_R(M, N) \otimes_R R' \cong \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R').$$

Now onto the proof of Proposition 1.1.2.

Proof. First of all, let's call $C := B/A$ and observe that $A \hookrightarrow B$ splits if and only if $B \twoheadrightarrow C$ splits, i.e. if the natural map $\mathrm{Hom}_A(C, B) \to \mathrm{Hom}_A(C, C)$ is surjective. This happens if and only if the localization of this map in $\mathfrak{p}$ is surjective for every prime ideal $\mathfrak{p} \subset A$.

B is module-finite over A, so C is a finitely generated A-module, and since A is noetherian C is also finitely presented: by the previous lemma, the functor $\mathrm{Hom}_A(C,\cdot)\otimes_A A_{\mathfrak{p}}$ and the functor $\mathrm{Hom}_{A_{\mathfrak{p}}}(C\otimes_A A_{\mathfrak{p}},\cdot\otimes_A A_{\mathfrak{p}})$ are naturally isomorphic. In particular, we have the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_A(C,B)\otimes_A A_{\mathfrak{p}} & \longrightarrow & \mathrm{Hom}_A(C,C)\otimes_A A_{\mathfrak{p}} \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{A_{\mathfrak{p}}}(C_{\mathfrak{p}},B_{\mathfrak{p}}) & \longrightarrow & \mathrm{Hom}_{A_{\mathfrak{p}}}(C_{\mathfrak{p}},C_{\mathfrak{p}}). \end{array}$$

The arrow above is surjective if and only if the arrow below is surjective, which happens if and only if the inclusion $A_p \hookrightarrow B_p$ splits.

For the second point, let's first observe that $\mathrm{Hom}_A(C,B)\to\mathrm{Hom}_A(C,C)$ is surjective if and only if $\mathrm{Hom}_A(C,B)\otimes_A A'\to\mathrm{Hom}_A(C,C)\otimes_A A'$ is surjective, because $A\hookrightarrow A'$ is faithfully flat. With the same reasoning as before we get to the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_A(C,B)\otimes_A A' & \longrightarrow & \mathrm{Hom}_A(C,C)\otimes_A A' \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Hom}_{A'}(C\otimes_A A',B\otimes_A A') & \longrightarrow & \mathrm{Hom}_{A'}(C\otimes_A A',C\otimes_A A'). \end{array}$$

and since the arrow below is surjective if and only if $A'\hookrightarrow B\otimes_A A'$ splits, the proposition is proven. □

As a further simplification, let's prove that we can assume the extension B to be a domain with the following proposition.

Proposition 1.1.7. *Let A be a domain and $i:A\to B$ be a module-finite ring extension. There is a prime $\mathfrak{q}\subset B$ (with projection $\pi_{\mathfrak{q}}:B\to B/\mathfrak{q}$) such that:*

1. *the composite map $\pi_{\mathfrak{q}}\circ i:A\to B/\mathfrak{q}$ is injective;*
2. *if $A\to B/\mathfrak{q}$ splits, then $A\to B$ splits.*

Vice versa, if $i:A\to B$ splits we can choose $\mathfrak{q}\subset B$ prime such that $\pi_{\mathfrak{q}}\circ i:A\to B/\mathfrak{q}$ is injective and splits.

Proof. Since A is a domain, $S:=i(A\setminus\{0\})$ is a multiplicatively closed set in B. Take a maximal ideal in $S^{-1}B$: it corresponds to a prime ideal $\mathfrak{q}\subset B$ with empty intersection with S. In particular, the composite map $\pi_{\mathfrak{q}}\circ i:A\to B/\mathfrak{q}$ is injective.

Moreover, if $r_{\mathfrak{q}}:B/\mathfrak{q}\to A$ is a retraction for $\pi_{\mathfrak{q}}\circ i$, we have $r_{\mathfrak{q}}\circ\pi_{\mathfrak{q}}\circ i=id_A$, thus $r_{\mathfrak{q}}\circ\pi_{\mathfrak{q}}$ is a retraction for $i:A\to B$.

Vice versa, if $r:B\to A$ is a retraction for i, set $\mathfrak{q}:=\ker(r)$ (which is prime because A is a domain): r factorizes as $r_{\mathfrak{q}}\circ\pi_{\mathfrak{q}}$, where $r_{\mathfrak{q}}:B/\mathfrak{q}\to A$ is the map induced on the quotient, and since $id_A=r\circ i=r_{\mathfrak{q}}\circ\pi_{\mathfrak{q}}\circ i$, the map $\pi_{\mathfrak{q}}\circ i:A\to B/\mathfrak{q}$ is injective with retraction $r_{\mathfrak{q}}$. □

By Proposition 1.1.2.i we can always assume A to be local. We now state concisely (without proof), in the case of mixed characteristic $(0,p)$, all the other assumptions that we can make about A and B without loss of generality.

Proposition 1.1.8. *The following statements are equivalent.*

- *For all regular local rings $(A,\mathfrak{m})$ of mixed characteristic $(0,p)$, every module-finite extension $A\hookrightarrow B$ splits as a map of A-modules.*

- *For all regular local rings $(A, \mathfrak{m})$ of mixed characteristic $(0, p)$, which are complete, unramified (i.e. $p \in \mathfrak{m} \setminus \mathfrak{m}^2$), and with $A/\mathfrak{m}$ algebraically closed, every module-finite integral extension $A \hookrightarrow B$ splits as a map of A-modules*

Remark 1.1.9. We just proved in Proposition 1.1.7 that we can assume the extension B to be a domain.

We can assume A to be complete and with algebraically closed residue field by replacing it with a suitable faithfully flat extension, as permitted by Proposition 1.1.2.ii.

The reason why we can assume A to be unramified is more complex, but an exhaustive proof can be found in [Hoc83, Theorem 6.1]. Bhatt has circumvented this problem in [Bha18, Proposition 5.2], but to provide a simpler construction in the last chapter, we will assume non-ramification.

1.2 Finite characteristic

In this section, we will prove the direct summand theorem in the case of positive equal characteristic p. The main idea for the proof of this particular case is borrowed from an observation of Hochster, which can be found in [Hoc83, Remark 6.2].

Theorem 1.2.1. *Let A be a regular ring of characteristic p, and let $A \hookrightarrow B$ be a module-finite extension of A. This inclusion splits as a map of A-modules.*

By Proposition 1.1.2 we can assume A to be local, and by what we said in Remark 1.1.9 we can also assume A to be complete with an algebraically closed residue field, and B to be a domain

Let's state a fundamental structure theorem about regular rings of finite characteristic:

Theorem 1.2.2 (Cohen)**.** *Let A be a complete regular local ring of finite characteristic p and dimension d, and let k be its residue field. Then A is isomorphic to a ring of power series on k:*

$$A \cong k[[x_1, \cdots, x_d]].$$

Before the proof of Theorem 1.2.1, let's consider an easy lemma.

Lemma 1.2.3. *There is a non zero map of A modules $\phi : B \to A$.*

Proof. Let's first prove that B can be embedded in a free A-module.

Let K and L be respectively the fraction fields of A and B: we have that B is contained in L, which is a finite dimensional K-vector space with basis $\{e_j\}_j$. If we consider a finite set of generators $\{b_i\}_i$ of B as an A-module, we can write $b_i = \sum_j \frac{s_{i,j}}{t_{i,j}} e_j$ with $s_{i,j} \in A$, $t_{i,j} \in A \setminus \{0\}$. Let $t := \prod_{i,j} t_{i,j} \neq 0$: B is contained in $F := \frac{1}{t}\mathrm{Span}_A(\{e_i\})$, which is a free module of finite rank.

Since $B \subseteq F$, at least one of the projections of B onto the coordinates of F must be not identically zero: this is a non zero map of A-modules from B to A. □

Now we can prove Theorem 1.2.1.

Proof. Consider a non zero map of A modules $\phi : B \to A$, as constructed in the previous lemma. Without loss of generality, we can assume $\phi(1) \neq 0$: if $b \in B$ is such that $\phi(b) \neq 0$, we could simply consider the A-linear map which sends $x \in B$ to $\phi(bx)$. Since A is complete with respect

to its maximal ideal $\mathfrak{m} := (x_1, \cdots, x_n)$, we have that $\bigcap_{k>0} \mathfrak{m}^k = 0$; in particular, there is some $e > 0$ such that $\phi(1) \notin \mathfrak{m}^{p^e}$.

If we call Φ the Frobenius endomorphism, we have that $\Phi^e(A) = k[[x_1^{p^e}, \cdots, x_d^{p^e}]]$, and the maximal ideal in $\Phi^e(A)$ is exactly $\Phi^e(\mathfrak{m})$. If we work modulo this ideal, since we have that $\Phi^e(\mathfrak{m})A = (x_1^{p^e}, \cdots, x_d^{p^e}) \subseteq \mathfrak{m}^{p^e}$, the projection of $\phi(1)$ on $A/\Phi^e(\mathfrak{m})A$ is not zero: this means that it can be completed to a basis of $A/\Phi^e(\mathfrak{m})A$ as a $\Phi(A)/\Phi(\mathfrak{m})$-module, and by Nakayama's lemma $\phi(1)$ is part of a minimal set of generators of A as a $\Phi^e(A)$-module. On the other hand, A is a free $\Phi^e(A)$-module of rank p^{ed}, since we can take as a basis the set of monomials in the variables $x_1, \cdots, x_n$ and exponent less than p^e. Thus, $\phi(1)$ is part of a free basis of A as a $\Phi^e(A)$-module, and we can consider the $\Phi^e(A)$-linear projection p from A to the component generated by $\phi(1)$ (this component can be identified with $\Phi^e(A)$ by sending the element $\phi(1)$ to $1 \in A$). If we call i the inclusion map $\Phi^e(B) \hookrightarrow B$, the composite map $\psi := p \circ \phi \circ i : \Phi^e(B) \to \Phi^e(A)$ is a $\Phi^e(A)$-linear map such that $\psi(1) = 1$, therefore it is a retraction of the inclusion map $\Phi^e(A) \hookrightarrow \Phi^e(B)$. Since Φ^e is injective, we get the following commutative diagram:

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow \cong & & \downarrow \cong \\ \Phi^e(A) & \hookrightarrow & \Phi^e(B), \end{array}$$

so the inclusion $A \hookrightarrow B$ also admits a retraction. □

1.3 Characteristic 0

In this section, we will prove the direct summand theorem in the case of equal characteristic 0. In this case we won't need regularity, and we can state the theorem as follows.

Theorem 1.3.1. *Let A be a domain of equal characteristic 0, and let $A \hookrightarrow B$ be a module-finite extension of A. This inclusion splits as a map of A-modules.*

We just need two easy lemmas.

Lemma 1.3.2. *A ring A has characteristic 0 if and only if it contains a copy of $\mathbb{Q}$.*

Proof. Let $i : \mathbb{Q} \hookrightarrow A$. Since every element of $\mathbb{Q} \setminus \{0\}$ has an inverse, $i(\mathbb{Q}) \cap \mathfrak{p} = \{0\}$ for every prime ideal $\mathfrak{p} \subset A$, therefore the composite map $\pi_{\mathfrak{p}} \circ i : \mathbb{Q} \to A/\mathfrak{p}$ is injective, and $A/\mathfrak{p}$ has characteristic 0.

On the other hand, if A has characteristic 0, there is an immersion $i : \mathbb{Z} \hookrightarrow A$, and since for every prime ideal $\mathfrak{p}$ $A/\mathfrak{p}$ has characteristic 0, it follows that $\mathfrak{p} \cap i(\mathbb{Z}) = \{0\}$. This means that every element in $i(\mathbb{Z}) \setminus \{0\}$ is invertible, therefore the inclusion i can be extended to $\mathbb{Q}$. □

Lemma 1.3.3. *Let $A \hookrightarrow B$ be a module-finite extension of domains. If A is a field, then B is a field.*

Proof. Fix any $b \in B \setminus A$. Since b is not a zero-divisor, it admits an irreducible minimal polynomial with coefficients in A: $\mu(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, for some $n > 1$. Since μ is irreducible $a_0 \neq 0$; if we call $f(x) := \frac{\mu(x) - a_0}{x}$, we get that $bf(b) = a_0$, and since a_0 is invertible, so is b. □

Corollary 1.3.4. *Let $A \hookrightarrow B$ is a module-finite extension of domains, with $S := A \setminus \{0\}$. Then $S^{-1}B$ is the fraction field L of B.*

Proof. $S^{-1}A \hookrightarrow S^{-1}B$ is a module-finite extension and $S^{-1}A$ is a field, so by the previous lemma $S^{-1}B$ is also a field; since L is the minimal field that admits an injection $B \hookrightarrow L$, we get an injection $L \hookrightarrow S^{-1}B$. On the other hand, obviously $S^{-1}B$ is contained in L, therefore they are equal. □

We are ready to prove Theorem 1.3.1.

Proof. As shown in Proposition 1.1.7, it suffices to show that any module-finite extension $A \hookrightarrow B$ with B domain splits. Let $S := A \setminus \{0\}$; since $A \hookrightarrow B$ is a module-finite extension, $S^{-1}B$ is the fraction field of B, and $S^{-1}A \hookrightarrow S^{-1}B$ is a finite extension of fields. Let $tr : S^{-1}B \to S^{-1}A$ be the trace map: for all $a \in A$ $tr(a) = na$, where $n := [S^{-1}B : S^{-1}A]$, and since $A \hookrightarrow B$ is an integral extension, $tr(B) \subseteq A$. Finally, n is invertible is A, thus $\frac{1}{n}tr : B \to A$ is a retraction of $A \hookrightarrow B$. □

Remark 1.3.5. This line of reasoning fails if A has mixed characteristic because n in general will not be invertible; on the other hand, we can still deduce that the localized map $A\left[\frac{1}{n}\right] \hookrightarrow B\left[\frac{1}{n}\right]$ splits. On its own, this result does not help because this localization is not faithful, but it can redirect us towards the solution: one may look for some theorems that from the splitting of the localization deduce the splitting of the original map. Such a result would be too strong to hope for, but there is a weaker version in a particular environment: Faltings' almost purity theorem. It will concern étale morphism (which is a slightly stronger condition than the splitting property), and its setting of almost mathematics will be explored in the next chapter.

Chapter 2

Almost mathematics

In this chapter we will give an introduction to almost mathematics, a theory first elaborated by Gerd Faltings to facilitate the study of some particular non-noetherian rings. All the concepts and proofs that are contained here can be found, explored in much greater detail and generality, in a comprehensive work from Gabber and Ramero: [GR03].

In the rest of the chapter, V will be a ring with a set element $t \in V$ which is not a zero divisor and admits a system of p-power roots for some prime number p; the ideal generated by all these roots will be called $I := (t^{\frac{1}{p^\infty}})$.

Example 2.0.1. Consider the ring $\mathbb{Z}_p$ of p-adic integers (that is the completion of $\mathbb{Z}$ with respect to the ideal p), with quotient field $\mathbb{Q}_p$. Recursively, take $x_0 := p$ and choose $\{x_n\}_n$ in the algebraic closure of $\mathbb{Q}_p$ such that $x_{n+1}^p = x_n$. If we add all the x_i to $\mathbb{Z}_p$ we get the ring $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$, with a system of p-power roots of p. This ring will be the building block for the construction of perfectoid theory, in Chapter 3.

2.1 The category of almost-modules

Starting from the category $V-\mathbf{Mod}$ of V-modules we want to work with some sort of "quotient" category, where we collapse to zero every "little" module, by which we mean every module which is annihilated after multiplication by any element in I. We will call it the category of *almost-V-modules*: $V^a - \mathbf{Mod}$. To describe this category, we first need to prove the following lemma:

Lemma 2.1.1. *Let V, I be as already defined. Then:*

- *as a V-module, I is flat;*
- *there is an equality $I^2 = I$;*
- *The multiplication map $m : I \otimes I \to I$ is an isomorphism.*

Proof.

- The ideal I is the increasing union of the ideals $I_k := (t^{\frac{1}{p^k}})$. Since I_k is principal for all k, and t is not a zero divisor, they are all isomorphic to V as V-modules, hence flat. Since filtered colimits of flat modules are flat, this implies that I is flat.
- Obviously $I^2 \subseteq I$. Viceversa, $t^{\frac{1}{p^k}}$ is a multiple of $t^{\frac{2}{p^{k+1}}} \in I^2$, therefore $I \subseteq I^2$.

- Consider the inclusion $I \hookrightarrow V$: tensoring by I we get the multiplication map $m : I \otimes I \to I$. It is injective because I is flat, but it is surjective by the previous point, therefore it is an isomorphism.

□

Now we can explicitly describe the category $\mathbf{Mod} - V^a$ in the following way:

- for every object $M \in V - \mathbf{Mod}$ there is an object $M^a \in V^a - \mathbf{Mod}$;
- the morphisms $\mathrm{Hom}_{V^a}(M^a, N^a)$ are identified with $\mathrm{Hom}_V(I \otimes M, N)$;
- the identity in $\mathrm{Hom}_{V^a}(M^a, M^a) \cong \mathrm{Hom}_V(I \otimes M, M)$ is the natural map induced by the inclusion $I \subseteq V$;
- given $f \in \mathrm{Hom}_{V^a}(M^a, N^a)$ and $g \in \mathrm{Hom}_{V^a}(N^a, P^a)$ the composition $g \circ^a f$ is defined as:

$$g \circ (id_M \otimes f) \circ (m^{-1} \otimes id_M) : I \otimes M \to I \otimes I \otimes M \to I \otimes N \to P.$$

Remark 2.1.2. The almost mathematics depends on the choice of the element t, but when this choice is clear from the context we will omit the dependence for clarity of exposition.

Remark 2.1.3. It is worth noting that we can take $t = 1$: in this case, the almost category is the same as the original category. This means that "usual" mathematics is properly a particular case of almost mathematics.

There is a natural functor $(-)^a : V - \mathbf{Mod} \to V^a - \mathbf{Mod}$, which sends M to M^a and $f : M \to N$ to the induced morphism $f^a : I \otimes M \to N$. The category $V^a - \mathbf{Mod}$ is abelian and inherits via this natural functor a notion of tensor product from the original category.

If we start with a V-algebra R with no t-torsion, we can similarly define $R^a - \mathbf{Mod}$; the morphisms will be:

$$\mathrm{Hom}_{R^a}(M^a, N^a) := \mathrm{Hom}_R(IR \otimes_R M, N).$$

Remark 2.1.4. This is the same as $\mathrm{Hom}_R(I \otimes_V M, N) \cong \mathrm{Hom}_R((I \otimes_V R) \otimes_R M, N)$. The reason is that for all k, since R has no t-torsion, we have the isomorphisms $I_k R \cong R \cong V \otimes_V R \cong I_k \otimes R$, where $I_k = (t^{\frac{1}{p^k}}) \subseteq V$; therefore, by passing to the colimit, we get $IR \cong I \otimes_V R$.

In particular, the different definitions of almost mathematics are compatible.

Remark 2.1.5. Since $V^a - \mathbf{Mod}$ has inherited a notion of tensor product, it is possible to define a V^a-algebra A as an algebra in the category $V^a - \mathbf{Mod}$; similarly we can also define a "module" over A: their category will be simply written as $A - \mathbf{Mod}$. It can be verified that, if $A = R^a$, the categories $A - \mathbf{Mod}$ and $R^a - \mathbf{Mod}$ are equivalent, so there is no ambiguity of definitions.

The main definition in the context of almost mathematics is that of (t-)almost zero module. Let's explore it with a lemma.

Lemma 2.1.6. *Take $M \in R - \mathbf{Mod}$. The following are equivalent:*

- *the R-module $IM = 0$;*
- *the R-module $I \otimes_V M = 0$;*
- *the object $M^a \in R^a - \mathbf{Mod}$ is the zero object.*

In this case, M is said to be t-almost zero, and we will write $M \approx_t 0$.

Proof. Let's prove (2) $\Leftrightarrow$ (3):

$$\begin{aligned} M^a \in R^a - \mathbf{Mod} \text{ is the zero object } &\Leftrightarrow \forall N \in R - \mathbf{Mod},\ \mathrm{Hom}_{R^a}(M^a, N^a) = 0 \\ &\Leftrightarrow \forall N \in R - \mathbf{Mod},\ \mathrm{Hom}_R(IR \otimes_R M, N) = 0 \\ &\Leftrightarrow IR \otimes_R M = 0 \\ &\Leftrightarrow I \otimes_V M = 0. \end{aligned}$$

Since IM is the image of the natural map $I \otimes_V M \to M$, (2) $\Rightarrow$ (1) is obvious. Vice versa, consider a generic element in $I \otimes_V M$: we just have to prove that it is zero. Since $I \subseteq V$ is an ideal, we can always write such an element as $1 \otimes x$, with $x \in M$. The map $I \otimes_V M \to IM$ is identically zero, so we get $x = m(1 \otimes x) = 0$, which means that $1 \otimes x = 0$. □

Having clarified this basic notion, we can study the relationship between the categories $R - \mathbf{Mod}$ and $R^a - \mathbf{Mod}$.

Proposition 2.1.7. *Let R be a V-algebra.*

- *The functor $(-)^a : R\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}$ has a right adjoint $(-)_* : R^a\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$. For $M \in R^a - \mathbf{Mod}$, with underlying R-module M_0, M_* is defined as follows:*

  $$M_* := \mathrm{Hom}_{R^a - \mathbf{Mod}}(R^a, M) \cong \mathrm{Hom}_{R - \mathbf{Mod}}(IR, M_0),$$

  while on the maps it is defined in the obvious way.

- *The counit $\varepsilon_M : (M_*)^a \to M$ (i.e. the map that corresponds to the identity $id_{M_*} : M_* \to M_*$ via the adjunction) is an isomorphism.*

- *The functor $(-)^a : R\text{-}\mathbf{Mod} \to R^a\text{-}\mathbf{Mod}$ has a left adjoint $(-)_! : R^a\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$. It is defined as the composition of the functor $(-)_*$ and the functor $- \otimes_V I$.*

  Proof. In this proof, tensor products will all be considered with respect to R.

  Let $M \in R^a - \mathbf{Mod}$, $N \in R - \mathbf{Mod}$, with M_0 as above:

  $$\begin{aligned} \mathrm{Hom}_{R - \mathbf{Mod}}(N, M_*) &= \mathrm{Hom}_{R - \mathbf{Mod}}(N, \mathrm{Hom}_{R - \mathbf{Mod}}(IR, M_0)) \\ &\cong \mathrm{Hom}_{R - \mathbf{Mod}}(IR \otimes N, M_0) \\ &\cong \mathrm{Hom}_{R^a - \mathbf{Mod}}(N^a, (M_0)^a) = \mathrm{Hom}_{R^a - \mathbf{Mod}}(N^a, M), \end{aligned}$$

  so the functors are adjoint.

- If we take $N := M_*$, this adjunction sends the identity map id_{M_*} to:

  $$\varepsilon_M \in \mathrm{Hom}_{R^a - \mathbf{Mod}}((M_*)^a, M) \cong \mathrm{Hom}_{R - \mathbf{Mod}}(IR \otimes \mathrm{Hom}_{R - \mathbf{Mod}}(IR, M_0), M_0),$$

  with $\varepsilon_M(a \otimes f) = f(a)$. Let's consider:

  $$\phi_M \in \mathrm{Hom}_{R^a - \mathbf{Mod}}(M, (M_*)^a) \cong \mathrm{Hom}_{R - \mathbf{Mod}}(IR \otimes M_0, \mathrm{Hom}_{R - \mathbf{Mod}}(IR, M_0)),$$

  such that $\phi_M(a \otimes x) = (b \to bax)$. Let's prove that it's the inverse of ε_M. In one direction:

$$IR \otimes IR \otimes \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0) \xrightarrow{id_{IR} \otimes \varepsilon_M} IR \otimes M_0 \xrightarrow{\phi_M} \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0)$$

$$a \otimes b \otimes f \longrightarrow a \otimes f(b) \longrightarrow (c \mapsto caf(b) = abf(c)),$$

therefore $\phi_M \circ^a \varepsilon_M$, viewed as an element of $\mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_*, M_*)$, sends $a \otimes f$ to af, which is the identity map in $R^a - \mathbf{Mod}$.

Vice versa:

$$IR \otimes IR \otimes M_0 \xrightarrow{id_{IR} \otimes \phi_M} IR \otimes \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0) \xrightarrow{\varepsilon_M} M_0$$

$$a \otimes b \otimes x \longrightarrow a \otimes (c \mapsto cbx) \longrightarrow abx,$$

therefore $\varepsilon_M \circ^a \phi_M$, viewed as an element of $\mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_0, M_0)$, sends $a \otimes x$ to ax, which is the identity map in $R^a - \mathbf{Mod}$.

- Let $M \in R^a - \mathbf{Mod}$, $N \in R - \mathbf{Mod}$, with M_0 as above:

$$\begin{aligned} \mathrm{Hom}_{R-\mathbf{Mod}}(M_!, N) &= \mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_*, N) \\ &\cong \mathrm{Hom}_{R^a-\mathbf{Mod}}((M_*)^a, N^a) \\ &\cong \mathrm{Hom}_{R^a-\mathbf{Mod}}(M, N^a), \end{aligned}$$

where in the last isomorphism we used that M and M_*^a are isomorphic in $R^a - \mathbf{Mod}$ via ε_M.

□

2.2 First definitions

In this new context we can give many definitions analogous to classical ones, and many properties are preserved - with appropriate alterations - in the new setting. The following examples are the ones we will need in this thesis. Let M, N be R-modules:

- the R-module M is almost zero ($M \approx 0$) if $M^a \in R^a - \mathbf{Mod}$ is the zero object, which happens if and only if M is I-torsion;
- the R-linear map $f : M \to N$ is almost injective (resp. almost surjective) if $\ker(f) \approx 0$ (resp. if $\mathrm{coker}(f) \approx 0$);
- the R-linear map $f : M \to N$ is an almost isomorphism if it is almost injective and almost surjective;
- the R-module M is almost projective (resp. almost flat) if for all $N \in R - \mathbf{Mod}$, the R-module $\mathrm{Ext}^1_R(M, N) \approx 0$ (resp. if $\mathrm{Tor}^R_1(M, N) \approx 0$);
- the R-module M is almost faithfully flat if it is almost flat and for all $N_1, N_2 \in R - \mathbf{Mod}$ the induced map $\mathrm{Hom}_R(N_1, N_2) \to \mathrm{Hom}_R(N_1 \otimes M, N_2 \otimes M)$ is almost injective;

- the sequence of R-modules $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ is said to be almost exact at N_2 if $\ker(g)/\operatorname{im}(f) \approx 0$.

Remark 2.2.1. It can be checked that the definition of almost isomorphism is compatible with the intrinsic notion of isomorphism in $R^a - \mathbf{Mod}$.

All the definitions we gave are invariant under almost isomorphism. We will just give an example, and check the following property.

Proposition 2.2.2. *Let M, N, P be R-modules, and $f : M \to N$ an almost isomorphism. The natural map $\operatorname{Ext}^k_R(P, M) \to \operatorname{Ext}^k_R(P, N)$ is an almost isomorphism for all $k \geq 0$.*

First, let's prove another lemma which highlights the good behaviour of almost zero modules.

Lemma 2.2.3. *Let $F : R - \mathbf{Mod} \to R - \mathbf{Mod}$ be an R-linear functor, K an almost zero R-module. The R-module $F(K)$ is almost zero.*

Proof. Since F is R-linear, it preserves the endomorphism $- \cdot t^\alpha$ for all $\alpha > 0$. On the module K this endomorphism is zero, therefore the same happens for $F(K)$; every element of $F(K)$ is t^α-torsion for all $\alpha > 0$, therefore $F(K) \approx 0$. □

Now we can prove the proposition.

Proof. We can write f as the composition $M \xrightarrow{p} \operatorname{im}(f) \xrightarrow{i} N$, where p is surjective and almost injective, while i is injective and almost surjective.

- Consider the exact sequence $0 \to K \to M \to \operatorname{im}(f) \to 0$, where $K \approx 0$. Applying the functor $\operatorname{Hom}_R(P, -)$ we get the following piece of a long exact sequence for all $k \geq 0$:

  $$\operatorname{Ext}^k_R(P, K) \to \operatorname{Ext}^k_R(P, M) \to \operatorname{Ext}^k_R(P, \operatorname{im}(f)) \to \operatorname{Ext}^{k+1}_R(P, K).$$

  By the previous lemma, since $K \approx 0$ and $\operatorname{Ext}^k_R(P, -)$ is an R-linear functor for all k, the first and last term of the exact sequence are almost zero, therefore the central map is an almost isomorphism

- Consider the exact sequence $0 \to \operatorname{im}(f) \to N \to C \to 0$, where $C \approx 0$. Applying the functor $\operatorname{Hom}_R(P, -)$ we get the following piece of a long exact sequence for all $k \geq 0$:

  $$\operatorname{Ext}^{k-1}_R(P, C) \to \operatorname{Ext}^k_R(P, \operatorname{im}(f)) \to \operatorname{Ext}^k_R(P, N) \to \operatorname{Ext}^k_R(P, C),$$

  where if $k = 0$ the functor $\operatorname{Ext}^{k-1}_R(P, -)$ is identically zero. By the previous lemma, since $C \approx 0$ and $\operatorname{Ext}^k_R(P, -)$ is an R-linear functor for all k, the first and last term of the exact sequence are almost zero, therefore the central map is an almost isomorphism.

Finally, we get that the composite map $\operatorname{Ext}^k_R(P, M) \to \operatorname{Ext}^k_R(P, \operatorname{im}(f)) \to \operatorname{Ext}^k_R(P, N)$ is an almost isomorphism for all $k \geq 0$. □

Remark 2.2.4. It's not obvious a priori that the definitions we gave are the most appropriate generalizations.

For example, we could define an almost projective module M as a module such that M^a is a projective object in the category $R^a - \mathbf{Mod}$. The problem of this definition would be that $R^a - \mathbf{Mod}$ does not have enough projectives (in particular, one could prove that R^a is not a projective object in $R^a - \mathbf{Mod}$).

Some reasons to think of those definitions as the "right" ones will be given in the next section.

2.3 Projectiveness, flatness and faithful flatness

The most convincing argument for the use of the previous definitions is to show that many "classical" properties are translated nicely into the language of the almost mathematics, for example the following:

Proposition 2.3.1. *Let M be an R-module.*

- *The following are equivalent:*
  1. *the R-module M is almost projective;*
  2. *for every surjective map $g : N_2 \to N_3$, for every map $\phi : M \to N_3$, for every $\alpha > 0$, the map $t^\alpha\phi$ factors through g;*
  3. *for every exact sequence of R-modules $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3) \to 0$ is almost exact.*
- *For every sequence of R-modules $0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ which is almost exact, the induced sequence $0 \longrightarrow \mathrm{Hom}_R(M, N_1) \xrightarrow{\tilde{f}} \mathrm{Hom}_R(M, N_2) \xrightarrow{\tilde{g}} \mathrm{Hom}_R(M, N_3)$ is almost exact. Moreover, if M is almost projective and g is almost surjective, $\tilde{g}$ is almost surjective.*

Proof. (1) implies (3) because we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, N_1) \longrightarrow \mathrm{Hom}_R(M, N_2) \longrightarrow \mathrm{Hom}_R(M, N_3) \longrightarrow \mathrm{Ext}^1_R(M, N_1),$$

where $\mathrm{Ext}^1_R(M, N_1) \approx 0$.

(3) implies (2) because since the map $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3)$ is almost surjective, $t^\alpha\phi$ is in its image for every $\phi \in \mathrm{Hom}_R(M, N_3)$ and for every $\alpha > 0$.

Now assume (2) is true. For any R-module N there is an injective module Q and an injective map $f : N \hookrightarrow Q$. Take the short exact sequence induced by f:

$$0 \longrightarrow N \xrightarrow{f} Q \xrightarrow{g} C \longrightarrow 0.$$

Since Q is injective, $\mathrm{Ext}^1_R(M, Q) = 0$ and we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{\tilde{f}} \mathrm{Hom}_R(M, Q) \xrightarrow{\tilde{g}} \mathrm{Hom}_R(M, C) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow 0.$$

For every map $\phi \in \mathrm{Hom}_R(M, C)$, for every $\alpha > 0$, the map $t^\alpha\phi$ factors through g, therefore $I \,\mathrm{Hom}_R(M, C) \subseteq \mathrm{im}(\tilde{g})$, and $\mathrm{Ext}^1_R(M, N) \cong \mathrm{coker}(\tilde{g}) \approx 0$.

For the second part, since $\ker(f)$ is almost zero, so is $\mathrm{Hom}_R(M, \ker(f)) \cong \ker(\tilde{f})$. For the exactness at N_2, we know that the inclusion $\mathrm{im}(f) \subseteq \ker(g)$ is almost surjective; for every $\phi \in \mathrm{Hom}_R(M, \ker(g))$, for every $\alpha > 0$, $t^\alpha\phi$ has image in $\mathrm{im}(f)$, therefore the inclusion

$$\mathrm{Hom}_R(M, \mathrm{im}(f)) \hookrightarrow \mathrm{Hom}_R(M, \ker(g)) \cong \ker(\tilde{g})$$

is almost surjective. In particular, if $N_3 = 0$ (i.e. if f is an almost isomorphism), $\tilde{f}$ is an almost isomorphism.

Now, suppose M is almost projective and g is almost surjective. Since $g : N_2 \to \mathrm{im}(g)$ is surjective, the induced map $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, \mathrm{im}(g))$ is almost surjective. Moreover, since $0 \to \mathrm{im}(g) \to N_3 \to 0$ is almost exact, $\mathrm{Hom}_R(M, \mathrm{im}(g)) \to \mathrm{Hom}_R(M, N_3)$ is an almost isomorphism, therefore the composition $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3)$ is almost surjective. □

Lemma 2.3.2. *Let P and Q be almost projective R-modules. The R-module $P \otimes Q$ is almost projective.*

Proof. Take an exact sequence of R-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since P and Q are almost projective, by 2.3.1.ii we get the following almost exact sequences:

$$0 \to \mathrm{Hom}_R(Q, N_1) \to \mathrm{Hom}_R(Q, N_2) \to \mathrm{Hom}_R(Q, N_3) \to 0;$$

$$0 \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_1)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_2)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_3)) \to 0.$$

Since the composite functor $\mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, -))$ is naturally isomorphic to $\mathrm{Hom}_R(P \otimes Q, -)$, by Lemma 2.3.1.i $P \otimes Q$ is an almost projective R-module. □

Lemma 2.3.3. *Let P_1 be a direct summand of an almost projective R-module P. The R-module P_1 is almost projective.*

Proof. Let $P \cong P_1 \oplus P_2$. Given a surjective map $f : M \twoheadrightarrow N$, let's call $\tilde{f} : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ the induced map. Since P is almost projective, the following modules are almost zero:

$$\frac{\mathrm{Hom}_R(P, N)}{\tilde{f}(\mathrm{Hom}_R(P, M))} \cong \frac{\mathrm{Hom}_R(P_1, N) \oplus \mathrm{Hom}_R(P_2, N)}{\tilde{f}(\mathrm{Hom}_R(P_1, M) \oplus \mathrm{Hom}_R(P_2, N))} \cong \frac{\mathrm{Hom}_R(P_1, N)}{\tilde{f}(\mathrm{Hom}_R(P_1, M))} \oplus \frac{\mathrm{Hom}_R(P_2, N)}{\tilde{f}(\mathrm{Hom}_R(P_2, M))}.$$

In particular $\mathrm{Hom}_R(P_1, N) \approx \tilde{f}(\mathrm{Hom}_R(P_1, M))$, so $\mathrm{Hom}_R(P_1, M)) \to \mathrm{Hom}_R(P_1, N)$ is almost surjective and P_1 is almost projective. □

Remark 2.3.4. Similar properties are true for almost flat modules, and the proofs are analogous.

Now we will show that almost-projectiveness and almost-flatness are related in a similar way as classical projectiveness and flatness.

Lemma 2.3.5. *Let P be an almost projective R-module. Then P is almost flat.*

Proof. Let C be any R-module and take an exact sequence $0 \to K \to F \to C \to 0$, where F is a free R-module. Tensoring by P we get the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}_R^1(C, P) \longrightarrow K \otimes P \longrightarrow F \otimes P \longrightarrow C \otimes P \longrightarrow 0,$$

where we used that $\mathrm{Tor}_R^1(F, P) = 0$ because F is free, and hence flat.

Let Q be an arbitrary injective module and apply the exact functor $\mathrm{Hom}_R(\cdot, Q)$:

$$0 \to \mathrm{Hom}_R(\mathrm{Tor}_R^1(C, P), Q) \to \mathrm{Hom}_R(K \otimes P, Q) \xrightarrow{f} \mathrm{Hom}_R(F \otimes P, Q) \xrightarrow{g} \mathrm{Hom}_R(C \otimes P, Q) \to 0.$$

From the properties of the tensor product, the functor $\mathrm{Hom}_R(\cdot \otimes P, Q)$ is naturally isomorphic to $\mathrm{Hom}_R(P, \mathrm{Hom}_R(\cdot, Q))$, composition of the functor $\mathrm{Hom}_R(P, \cdot)$, which is "almost" exact, and $\mathrm{Hom}_R(\cdot, Q)$, which is exact. This means that applying it to $0 \to K \to F \to C \to 0$ we get an exact sequence:

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(K, Q)) \xrightarrow{f'} \mathrm{Hom}_R(P, \mathrm{Hom}_R(F, Q)) \xrightarrow{g'} \mathrm{Hom}_R(P, \mathrm{Hom}_R(C, Q)) \longrightarrow 0,$$

with the added property that f' is almost injective.

The natural isomorphism of the two functors sends f to f' and g to g', therefore f is almost injective and $\mathrm{Hom}_R(\mathrm{Tor}_R^1(C,P),Q) \approx 0$.

We can choose Q such that there is an injective map $i : \mathrm{Tor}_R^1(C,P) \hookrightarrow Q$. Since the module $\mathrm{Hom}_R(\mathrm{Tor}_R^1(C,P),Q)$ is almost zero, $t^\alpha i$ is the zero map for every $\alpha > 0$. It follows that $t^\alpha \mathrm{Tor}_R^1(C,P) = 0$ for every $\alpha > 0$, i.e. $\mathrm{Tor}_R^1(C,P) \approx 0$. □

Proposition 2.3.6. *Let P be a finitely presented, almost flat R-module. Then P is almost projective.*

Proof. Fix an injective R-module Q, and call $\widetilde{-} : R\text{−}\mathbf{Mod} \to R\text{−}\mathbf{Mod}$ the functor $\widetilde{\mathrm{Hom}_R(-,Q)}$. For any R-modules M, N, there is a natural morphism $\alpha_{M,N} : \widetilde{N} \otimes M \to \widetilde{\mathrm{Hom}_R(M,N)}$ such that $\alpha_{M,N}(f \otimes x)$ is the map that sends $g \in \mathrm{Hom}_R(M,N)$ to $f(g(x))$. Let's prove that if M is finitely presented $\alpha_{M,N}$ is an isomorphism. If $M \cong R^m$ for some finite m, we have the following chain of isomorphism:

$$\widetilde{N} \otimes_R R^m \cong \mathrm{Hom}_R(N,Q)^m \cong \mathrm{Hom}_R(N^m,Q) \cong \widetilde{N^m} \cong \widetilde{\mathrm{Hom}_R(R^m,N)},$$

and the composite isomorphism is $\alpha_{R^m,N}$.

In general, if M is finitely presented, there is an exact sequence $R^k \to R^n \to M \to 0$. The functors $\widetilde{N} \otimes -$ and $\widetilde{\mathrm{Hom}_R(-,N)}$ are both right exact, so we get the following diagram:

$$\begin{array}{ccccccc}
\widetilde{N} \otimes R^k & \longrightarrow & \widetilde{N} \otimes R^n & \longrightarrow & \widetilde{N} \otimes M & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_{R^k,N}} & & \downarrow{\scriptstyle \alpha_{R^n,N}} & & \downarrow{\scriptstyle \alpha_{M,N}} & & \| \\
\widetilde{\mathrm{Hom}_R(R^k,N)} & \longrightarrow & \widetilde{\mathrm{Hom}_R(R^n,N)} & \longrightarrow & \widetilde{\mathrm{Hom}_R(M,N)} & \longrightarrow & 0.
\end{array}$$

The diagram commutes because α is a natural transformation, and since $\alpha_{R^n,N}$ and $\alpha_{R^k,N}$ are both isomorphism, $\alpha_{M,N}$ is also an isomorphism by the five lemma.

Now onto the main part of the proof. Let M be any R-module and take an injective R-module J so that we have a short exact sequence:

$$0 \longrightarrow M \longrightarrow J \longrightarrow C \longrightarrow 0.$$

Let P be a finitely presented, almost flat R-module. If we apply the functor $\mathrm{Hom}_R(P,-)$, and then the exact functor $\mathrm{Hom}_R(-,Q)$, since J is injective we get the following exact sequence:

$$0 \longrightarrow \widetilde{\mathrm{Ext}_R^1(P,M)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,C)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,J)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,M)} \longrightarrow 0.$$

If instead we apply the exact functor $\mathrm{Hom}_R(-,Q)$ and then the functor $- \otimes P$, we get an exact sequence:

$$\mathrm{Tor}_R^1(\widetilde{M},P) \longrightarrow \widetilde{C} \otimes P \longrightarrow \widetilde{J} \otimes P \longrightarrow \widetilde{M} \otimes P \longrightarrow 0,$$

where the leftmost module is almost zero because P is almost flat. Since α is a natural isomorphism, the rightmost parts of these sequences are isomorphic. This implies that $\widetilde{\mathrm{Ext}_R^1(P,M)}$ is an almost zero module. We can choose Q such that there is an injective map $i : \mathrm{Ext}_R^1(P,M) \to Q$: this map must be zero when multiplied by t^α for all α; in particular, its image, which is isomorphic to $\mathrm{Ext}_R^1(P,M)$, must be I-torsion, therefore almost zero.

By varying M, we get that P is an almost projective R-module. □

Let's now reexamine the concept of faithful flatness in the almost sense.

Proposition 2.3.7. *Let P be an almost flat R-module. The following are equivalent:*

- *as an R-module, P is almost faithfully flat;*
- *for every R-module M, $M \approx 0$ if and only if $M \otimes P \approx 0$;*
- *for every ideal $J \subseteq R$, if $R/J \not\approx 0$, then $P/JP \not\approx 0$.*

Proof. Assume (2) and take $f : M \to N$ such that the induced map $\tilde{f} : M \otimes P \to N \otimes P$ is zero. By the right exactness of tensor product we get $0 = \mathrm{im}(\tilde{f}) = \mathrm{im}(f) \otimes P$, therefore $\mathrm{im}(f) \approx 0$, i.e. the map f is almost zero.

Assume (1) and take M such that $M \otimes P \approx 0$. By almost faithful flatness, we have an almost injective map $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(P, M \otimes P) \approx 0$, therefore $M \cong \mathrm{Hom}_R(R, M) \approx 0$.

(2) obviously implies (3) by taking $M := A/J$. Vice versa, if $M \otimes P \approx 0$, choose $x \in M$ and consider the submodule generated by x, which is isomorphic to $R/\mathrm{Ann}(x)$. Since P is almost flat, the inclusion $R/\mathrm{Ann}(x) \hookrightarrow M$ induces an almost injective map $R/\mathrm{Ann}(x) \otimes P \to M \otimes P \approx 0$, therefore $R/\mathrm{Ann}(x) \approx 0$ by our hypothesis. Since x is arbitrary, this means every element of M is killed by multiplication with t^α for all α, i.e. $M \approx 0$. □

Another way to work with almost faithfully flat modules is to use the evaluation map, as shown in the following lemma.

Lemma 2.3.8. *Let R be an A-algebra, P a (IR-)almost projective R-module. Consider the natural map $ev_{P/R} : P \otimes \mathrm{Hom}_R(P, R) \to R$ given by the evaluation, and call $\mathcal{E}_{P/R}$ the image of this map. The following statements are true:*

- *for every morphism of algebras $R \to R'$, if $P' := P \otimes_R R'$, we have $\mathcal{E}_{P'/R'} \approx \mathcal{E}_{P/R}R'$;*
- *the ideal $\mathcal{E}_{P/R} \approx 0$ if and only if $P \approx 0$;*
- *the ideal $\mathcal{E}_{P/R} \approx R$ if and only if P is almost faithfully flat.*

Proof. • Consider a free module $F := R^{(J)}$ with a surjective map $\phi : F \to P$. For any $j \in J$, denote by $p_j : F \to R$ the projection on the j-th component, and with $e_j : R \to F$ the inclusion on the j-th component, so that $p_i \circ e_j = \delta_{ij} id_R$ and $\sum_j e_j \circ p_j = id_F$ (this sum is finite because for every $x \in F$ there is only a finite number of j such that $p_j(x) \neq 0$). Since P is almost projective, we make use of its lifting property, and take for every $\alpha > 0$ $\psi_\alpha \in \mathrm{Hom}_R(F, P)$ such that $\phi \circ \psi_\alpha = t^\alpha id_P$. Let's prove the following:

$$I\mathcal{E}_{P/R} \subseteq \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R \subseteq \mathcal{E}_{P/R},$$

Every generator of the center ideal can be obtained as $ev_{P/R}(\phi \circ e_j(1), p_i \circ \psi_\alpha)$.

On the other hand, take $f : P \to A$ and $x \in P$. Since ϕ is surjective, we can write $x = \phi(\sum_j a_j e_j(1))$, where $a_j \in A$ are all zero except for a finite number of indices. For every $\alpha > 0$ we have:

$$t^\alpha f = (f \circ \phi) \circ \psi_\alpha = f \circ \phi \circ \left(\sum_i e_i \circ p_i \right) \circ \psi_\alpha = \sum_i (f \circ \phi \circ e_i) \circ p_i \circ \psi_\alpha,$$

where $f \circ \phi \circ e_i : R \to R$ can be thought of as an element $b_i \in R$. Putting the formulas together we get that $t^\alpha f(x) = \sum_{i,j} a_j b_i p_i \circ \psi_\alpha \circ \phi \circ e_j(1)$, and since α, f and x are arbitrary, we get:

$$I\mathcal{E}_{P/R} \subseteq \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R \subseteq \mathcal{E}_{P/R},$$

which proves our claim.

Given a morphism of algebras $R \to R'$, with $P' := P \otimes_R R'$, we have:

$$\mathcal{E}_{P'/R'} \approx \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_{R'} \approx \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R R' \approx \mathcal{E}_{P/R} R'.$$

- For the second point, the left implication is obvious. Vice versa, if $\mathcal{E}_{P/R} \approx 0$, for every i, j, α the element $p_i \circ \psi_\alpha \circ \phi \circ e_j(1)$ is almost zero. Like before, take any $x \in P$ and write it as $\sum_j a_j \phi \circ e_j(1)$. We have:

$$0 \approx \sum_{i,j} a_j \phi \circ e_i \circ (p_i \circ \psi_\alpha \circ \phi \circ e_j(1)) = \sum_j a_j \phi \circ \psi_\alpha \circ \phi \circ e_j(1) = t^\alpha \sum_j b_j \phi \circ e_j(1) = t^\alpha x,$$

therefore $t^\alpha x$ is I-torsion. Since x and α were arbitrary, every element in P is I-torsion, i.e. $P \approx 0$.

- For every ideal $J \subseteq R$ such that $R/J \not\approx 0$ (which happens if and only if $IR \not\subseteq J$) we have:

$$P/JP (\cong P \otimes R/J) \not\approx 0 \iff \mathcal{E}_{\frac{P}{JP}/\frac{R}{J}} (\approx \mathcal{E}_{P/R} R/J) \not\approx 0 \iff I\mathcal{E}_{P/R} \not\subseteq J.$$

Remembering that by Lemma 2.3.5 P is almost flat, Lemma 2.3.7 tells us that the left hand side is true (for all J such that $IR \not\subseteq J$) if and only if P is almost faithfully flat. If the right hand side is true for all J, J can't be equal to $I\mathcal{E}_{P/R}$, therefore $IR \subseteq I\mathcal{E}_{P/R}$; vice versa, if $IR \subseteq I\mathcal{E}_{P/R}$, obviously the right hand side is true for all J which don't contain IR.

To wrap up, we have proven that P is almost faithfully flat if and only if $IR \subseteq I\mathcal{E}_{P/R}$, i.e. if and only if $\mathcal{E}_{P/R} \approx R$.

□

2.4 Ramification and étaleness

In this section we will talk about unramified and étale extensions, and we will show the proper way to translate these concepts into the language of almost mathematics, using the notions we already explored in the previous sections.

2.4.1 The module of Kähler differentials

We will remind the construction of the module of Kähler differentials and some of its basic properties. All the relative proofs can be found in [Sta21, Section 10.131].

For the rest of this subsection, A and B will be commutative rings.

Definition 2.4.1. Let $f : A \to B$ be a ring map and M a B-module. An *A-derivation* of B into M is an A-linear map $d : B \to M$ that satisfies the Leibniz rule: for every $b_1, b_2 \in B$, $d(b_1 b_2) = b_1 d(b_2) + b_2 d(b_1)$. The set of A-derivations of B into M is called $\mathrm{Der}_A(B, M)$.

Remark 2.4.2. The B-module structure on M induces a B-module structure on $\mathrm{Der}_A(B, M)$. Moreover, any map of B-modules $M \to N$ induces, via the composition, a B-linear map $\mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, N)$. It's easy to see that this implies that the derivations determine a functor $\mathrm{Der}_A(B, -) : B - \mathbf{Mod} \to B - \mathbf{Mod}$.

Proposition 2.4.3. *Let $f : A \to B$ be a ring map.*

- *The functor $Der_A(B, -) : B - \mathbf{Mod} \to B - \mathbf{Mod}$ is representable, which means that there is a B-module $\Omega_{B/A}$ and a derivation $d : B \to \Omega_{B/A}$ such that the natural transformation $\mathrm{Hom}_B(\Omega_{B/A}, -) \Rightarrow Der_A(B, -)$ which sends f to $f \circ d$ is an isomorphism.*
- *Consider the multiplication map $m : B \otimes_A B \to B$ and call I its kernel. There is an isomorphism of B-modules $\Omega_{B/A} \cong I/I^2$.*

Proof. We give a sketch of the proof.

We can prove both statements together by finding a derivation $d : B \to I/I^2$ such that I/I^2 and d satisfy the first point. First, we have to check that I/I^2 has a natural structure of B-module. It comes from the following isomorphism:

$$I/I^2 \cong I \otimes_{B\otimes_A B} (B \otimes_A B)/I \cong I \otimes_{B\otimes_A B} B,$$

and it can be easily checked that, for every $x \in I/I^2$ and $b \in B$, we have $b \cdot i = (b\otimes 1)\cdot i = 1\otimes(b\cdot i)$.

Let's define d as the composition of the map from B to I that sends an element b to $b\otimes 1 - 1\otimes b$, and the projection onto I/I^2: we want to prove that it is a derivation. Obviously d is A-linear, so we just have to check the Leibniz rule:

$$d(b_1b_2) = \overline{(b_1b_2)\otimes 1 - 1\otimes(b_1b_2)} = \overline{(b_1b_2)\otimes 1 - b_1\otimes b_2} + \overline{b_1\otimes b_2 - 1\otimes(b_1b_2)} = b_1 d(b_2) - d(b_1)b_2.$$

Now we can prove that the map $- \circ d$ provides the following isomorphism for every B-module M:

$$\mathrm{Hom}_B(I/I^2, M) \Rightarrow \mathrm{Der}_A(B, M).$$

We will just give a description of the inverse map, without checking that it has all the necessary properties. Take a derivation $\delta : B \to M$; via the properties of the tensor product, we can get two maps $\delta_1, \delta_2 : B \otimes_A B \to M$, the first B-linear on the left and the second B-linear on the right, such that $\delta_1(b_1 \otimes b_2) = b_1\delta(b_2)$ and $\delta_2(b_1 \otimes b_2) = \delta(b_1)b_2$. Their sum, restricted to I, factors through the quotient I/I^2: we call this function $\tilde{\delta}$; moreover, it's easy to check that $\tilde{\delta}$ is B-linear and that $\delta \to \tilde{\delta}$ is the inverse map to $- \circ d$. □

Definition 2.4.4. The module $\Omega_{B/A}$ is called the *module of Kähler differentials* of B over A.

2.4.2 Unramified morphisms

We will now give two alternative definitions of unramified morphism: the first will use the definition of the module of differentials, while the second will allow us to generalize this concept to almost mathematics.

Definition 2.4.5 (Definition 1)**.** A module-finite ring map $A \to B$ is said to be *unramified* if $\Omega_{B/A} = 0$.

Definition 2.4.6 (Definition 2)**.** A module-finite ring map $A \to B$ is said to be *unramified* if B is projective as a $B \otimes_A B$-module, via the multiplication map $m : B \otimes_A B \to B$.

Proposition 2.4.7. *The previous definitions are equivalent.*

Proof. By Proposition 2.4.3, if $\Omega_{B/A} = 0$, $I := \ker(m) = \ker(m)^2$. To describe I more explicitly, let's prove the following:

$$I = \mathrm{Span}_{B\otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}).$$

One inclusion is obvious. Vice versa take an element $x \in I$ with $x = \sum_j b'_j \otimes b''_j$. For every j, we can add the element $b'_j \otimes 1(b''_j \otimes 1 - 1 \otimes b''_j)$, which is contained on the right hand side; we get:

$$y := \sum_j \left(b'_j \otimes b''_j + (b'_j b''_j \otimes 1 - b'_j \otimes b''_j)\right) = \sum_j b'_j b''_j \otimes 1 = \left(\sum_j b'_j b''_j\right) \otimes 1.$$

Since $y \in I$, it's in the kernel of the multiplication map, so $\sum_j b'_j b''_j = 0$, therefore $y = 0$ and x is contained in the left hand side.

Fixed a set $\{b_1, \cdots, b_k\}$ of A-generators for B, we have that

$$\mathrm{Span}_{B\otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}) = \mathrm{Span}_{B\otimes_A B}(\{b_i \otimes 1 - 1 \otimes b_i\}_i),$$

which means that I is finitely generated as a $B\otimes_A B$-module. Since $I^2 = I$, by Nakayama's lemma we deduce that there is an element $e \in 1 + I$ such that $eI = 0$. If we write $e = 1 + i$, we get that $e^2 = e(1+i) = e + ei = e$, i.e. e is an idempotent. Obviously $B \otimes_A B = (e)_{B\otimes_A B} + (1-e)_{B\otimes_A B}$; moreover, the two submodules are direct summands because, if $x = ey = (1-e)z$ is in their intersection, $x = ex + (1-e)x = e(1-e)z + (1-e)ey = (e-e^2)z + (e-e^2)y = 0$. Since $eI = 0$, this decomposition implies $I \subseteq (1-e)B \otimes_A B$ so $I \cap eB \otimes_A B = 0$. Moreover, $m(e) = m(1+i) = 1$, so m induces a surjective map of $B \otimes_A B$-modules $eB \otimes_A B \to B$: since the kernel of this map is contained in I, it must be 0, therefore $B \cong eB \otimes_A B$ is a direct summand of $B \otimes_A B$, hence projective as a $B \otimes_A B$-module.

Viceversa, if B is projective as a $B \otimes_A B$-module, the multiplication map m induces a decomposition $B \otimes_A B = B \oplus I$, where $I = \ker(m)$. If we identify B with this ideal of $B \otimes_A B$, the multiplication map acts as a projection: on one hand $B = m(1)B \otimes_A B$, while on the other hand, since m is a projection and is $B \otimes B$-linear, we get that $m(1) = m(m(1)) = m(1 \cdot m(1)) = m(1) \cdot m(1)$. Moreover, $m(1)I = m(I) = 0$: with the same reasoning as the previous point we get:

$$I = (1 - m(1))B \otimes_A B \Longrightarrow I^2 = (1 - m(1))^2 B \otimes_A B = (1 - m(1))B \otimes_A B = I,$$

therefore, by Proposition 2.4.3, $\Omega_{B/A} \cong I/I^2 = 0$. □

We may now define finite étale coverings:

Definition 2.4.8. Let $A \to B$ be a module-finite ring map. It is said to be an étale covering if the following conditions are verified:

- as an A-module, B is faithfully flat;
- the extension $A \to B$ is unramified.

Remark 2.4.9. The usual definition only requires étale extensions to be flat on the base ring. However, when the extension is module-finite - like in our definition - the going up theorem proves that the induced map on the spectra $\mathrm{Spec}B \to \mathrm{Spec}A$ is surjective: this condition, together with flatness, implies faithful flatness.

We will now include a useful result, that any suitable module-finite extension of rings admits a localization which is a finite étale covering.

Proposition 2.4.10. *Let $A \to B$ be a module-finite separable extension of noetherian domains (with quotient fields K and L respectively). There is some $g \in A$, which we called the discriminant, such that $A[g^{-1}] \to B[g^{-1}]$ is étale.*

Proof. Remember that $A[g^{-1}] \to B[g^{-1}]$ is étale if B is a finitely generated, projective, and almost flat A-module, and it is projective as a $B[g^{-1}] \otimes_{A[g^{-1}]} B[g^{-1}] (\cong B \otimes_A B[g^{-1}])$-module. Let's divide the proof in two parts:

- There is some $f \in A$ such that $A[f^{-1}] \to B[f^{-1}]$ is free.

  Since the extension $K \to L$ is finite and separable, there is some $\alpha \in L$ such that:

  $$L = K(\alpha) = \bigoplus_{i=0}^{n-1} K\alpha^i,$$

  where $n = [L : K]$; up to multiplying α by a factor in K, we can assume $\alpha \in B$, whith monic polynomial μ of degree n. We have an inclusion of A-modules $i : A \oplus A\alpha \oplus \cdots \oplus A\alpha^{n-1} \to B$, which becomes an isomorphism after tensorization by the flat A-module K; in particular, if $C := \text{coker}(i)$, $C \otimes_A K = 0$. Since B is a finitely generated A-module, so is C, and we can take a finite set of generators $\{c_1, \cdots, c_k\}$; for every c_i, the corresponding element $c_i \otimes 1 \in C \otimes_A K$ is zero, which means that there is some $a_i \in A \setminus \{0\}$ such that $a_i c_i = 0$. Call f the product of the a_i's: since it kills all the generators of C, it belongs to $Ann_A(C)$, therefore if we invert f the map i becomes an isomorphism, i.e.:

  $$B[f^{-1}] = \bigoplus_{i=0}^{n-1} A[f^{-1}]\alpha^i.$$

- There is some $h \in A$ such that $B \otimes_A B[h^{-1}] \to B[h^{-1}]$ is projective.

  Call $I \subseteq B \otimes_A B$ the kernel of the multiplication map $m : B \otimes_A B \to B$. Fixed a set $\{b_1, \cdots, b_k\}$ of A-generators for B, in the proof of 2.4.7 we showed that:

  $$I = \text{Span}_{B \otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}) = \text{Span}_{B \otimes_A B}(\{b_i \otimes 1 - 1 \otimes b_i\}_i).$$

  Consider the natural map $\tilde{m} : \text{Hom}_{B \otimes_A B}(B, B \otimes_A B) \to \text{Hom}_{B \otimes_A B}(B, B)$: we just need to prove that, this map becomes surjective after inverting some $h \in A$. It's easy to show that the second module is simply B: on one hand there is an inclusion (of B-modules) $\text{Hom}_{B \otimes_A B}(B, B) \subseteq \text{Hom}_B(B, B) \cong B$, while on the other hand every B-linear morphism from B to B is automatically $B \otimes_A B$-linear. Now, take $\phi \in \text{Hom}_{B \otimes_A B}(B, B \otimes_A B)$; for all $b \in B$ we have:

  $$\phi(b) = (b \otimes 1)\phi(1) = (1 \otimes b)\phi(1) \Longrightarrow \phi(1)(b \otimes 1 - 1 \otimes b) = 0,$$

  so $\phi(1) \in \text{Ann}_{B \otimes_A B}(I)$. Vice versa, for the same reason, for all $x \in \text{Ann}_{B \otimes_A B}(I)$ the map $\phi(b) := (b \otimes 1)x$ is $B \otimes_A B$-linear, so we have that $\text{Hom}_{B \otimes_A B}(B, B \otimes_A B) \cong \text{Ann}_{B \otimes_A B}(I)$. Via this identifications, we can think of $\tilde{m}$ as the multiplication map from $\text{Ann}_{B \otimes_A B}(I)$ to B.

  This map's image is an ideal of B: let's prove that it is not zero. For every b_i, for all $k \geq 0$, define

  $$b_i^{(k)} := \sum_{j=0}^{k-1} b_i^j \otimes b_i^{k-1-j}$$

(so that $b_i^{(0)} = 0$), and note that

$$b_i^{(k)}(b_i \otimes 1 - 1 \otimes b_i) = b_i^k \otimes 1 - 1 \otimes b_i^k.$$

If $\mu_i(x) := \sum_k a_k x^k$ is the minimal polynomial of b_i with coefficients in A, we can consider the element $\tilde{b}_i := \sum_k a_k b_i^{(k)}$; we have that:

$$\tilde{b}_i(b_i \otimes 1 - 1 \otimes b_i) = \sum a_k b_i^{(k)}(b_i \otimes 1 - 1 \otimes b_i) = \sum a_k(b_i^k \otimes 1 - 1 \otimes b_i^k) = \mu_i(b_i) \otimes 1 - 1 \otimes \mu_i(b_i) = 0.$$

This means that the product $\tilde{b} := \prod_i \tilde{b}_i$ is in $\mathrm{Ann}_{B \otimes_A B}(I)$. Moreover, we have the equality $m(b_i^{(k)}) = k b_i^{k-1}$ and m is a ring homomorphism, so we get that:

$$m(\tilde{b}) = \prod_i m(\tilde{b}_i) = \prod_i m\left(\sum_k a_k b_i^{(k)}\right) = \prod_i \left(\sum_k a_k k b_i^{k-1}\right) = \prod \mu_i'(b_i),$$

and since B is a separable extension of A, for all i $\mu_i'(b_i) \neq 0$, and so is their product.

The inverse of $m(\tilde{b})$ is in L, therefore there is some $h \in A \setminus \{0\}$ such that $\frac{h}{m(\tilde{b})} \in B$, which means that $m(\tilde{b})$ divides h. In particular, if we invert h, the localization of the map $\tilde{m}$, whose image contains $m(\tilde{b})$, becomes surjective, and therefore $B[h^{-1}]$ is projective as a $B \otimes_A B[h^{-1}]$-module.

To conclude, we just have to take $g := hf$ so that by inverting g both of the previous conditions are satisfied. □

Remark 2.4.11. The hypothesis of separability is satisfied in particular when the quotient fields of A and B are of characteristic 0.

Example 2.4.12. To show that separability really is a necessary condition, let's consider the following extension of fields:

$$\mathbb{F}_p(t) \hookrightarrow \mathbb{F}_p(t^{\frac{1}{p}}).$$

This extension is inseparable, since $t^{\frac{1}{p}}$ has no conjugate roots other than itself. Moreover, since $\mathbb{F}_p(t)$ is a field, it has no nontrivial localizations, so we just need to show that this extension is ramified. We have:

$$\mathbb{F}_p(t^{\frac{1}{p}}) \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p(t^{\frac{1}{p}}) \cong \frac{\mathbb{F}_p(t)[x]}{(x^p - t)} \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p(t^{\frac{1}{p}}) \cong \frac{\mathbb{F}_p(t^{\frac{1}{p}})[x]}{(x^p - t)} \cong \frac{\mathbb{F}_p(t^{\frac{1}{p}})[x]}{(x - t^{\frac{1}{p}})^p}.$$

If we take $f(x) \in \mathbb{F}_p(t^{\frac{1}{p}})[x]$ such that its projection is idempotent, we get that $(x - t^{\frac{1}{p}})$ divides $f(x)^2 - f(x) = f(x)(f(x) - 1)$, so it must divide only one of the two factors, since they are coprime. This means that $f(x)$'s projection is either 0 or 1, thus the ring $\mathbb{F}_p(t^{\frac{1}{p}}) \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p(t^{\frac{1}{p}})$ does not have nontrivial idempotents, which by the proof of Proposition 2.4.7 means that $\mathbb{F}_p(t^{\frac{1}{p}})$ is ramified over $\mathbb{F}_p(t)$.

2.4.3 Almost finite étale coverings

For the rest of the chapter, let R, t, I be as in the previous section, and consider almost mathematics with respect to t.

After we laid down the groundwork in the previous subsection, we can properly generalize the definitions of non-ramification and étaleness.

Definition 2.4.13. Let $R \to S$ be a module-finite ring map. It is said to be almost unramified if S is almost projective as an $S \otimes_R S$-module.

Definition 2.4.14. Let $R \to S$ be a module-finite ring map. It is said to be an almost étale covering if the following conditions are verified:

- as an R-module, S is almost faithfully flat;
- the extension $R \to S$ is almost unramified.

Throughout the rest of this thesis, when we talk about almost finite étale maps, we will be referring to the previous definition.

Let's note with the following proposition that S is more than almost faithfully flat.

Proposition 2.4.15. *Let $R \to S$ be a module-finite ring map, which is almost étale. Then S is an almost projective R-module.*

Proof. Since S is finitely generated as an R-module, so is $S \otimes_R S$. Moreover, as seen in the proof of Proposition 2.4.7, the kernel I of the multiplication map is finitely generated as an $S \otimes_R S$-module, therefore it is also finitely generated as an R-module. This means that the exact sequence $I \to S \otimes_R S \to S \to 0$ makes S a finitely presented R-module: by Proposition 2.3.6, being almost flat, it is also almost projective as an R-module. □

Proposition 2.4.16. *Let $f : R \hookrightarrow S$ a module-finite ring map which is an almost étale covering. Then $C := \operatorname{coker}(f)$ is an almost projective R-module.*

Proof. Tensoring by S the exact sequence $0 \to R \to S \to C \to 0$ we get:

$$S \xrightarrow{\tilde{f}} S \otimes S \longrightarrow C \otimes S \longrightarrow 0,$$

where $\tilde{f}(b) = b \otimes 1$. The multiplication map $m : S \otimes S \to S$ sends $b_1 \otimes b_2$ to $b_1 b_2$, therefore $m \circ \tilde{f} = id_S$: not only $\tilde{f}$ is injective, but the sequence splits and we can write $S \otimes S \cong S \oplus C \otimes S$ as S-modules.

$S \otimes S$ is an almost projective S-module by Lemma 2.3.2, and $C \otimes S$ is almost projective by Lemma 2.3.3 because it is a direct summand of $S \otimes S$.

Let M be any R-module, and take a short exact sequence $0 \to M \to Q \to K \to 0$, where Q is an injective module. Since Q is injective, $\operatorname{Ext}^1_R(C, Q) = 0$, so if we apply the functor $\operatorname{Hom}_R(C, \cdot)$ we get the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}_R(C, M) \longrightarrow \operatorname{Hom}_R(C, I) \longrightarrow \operatorname{Hom}_R(C, K) \longrightarrow \operatorname{Ext}^1_R(C, M) \longrightarrow 0.$$

Since S is almost projective, applying the functor $\operatorname{Hom}_R(S, \cdot)$ we get an almost exact sequence:

$$0 \to \operatorname{Hom}_R(S \otimes C, M) \to \operatorname{Hom}_R(S \otimes C, Q) \to \operatorname{Hom}_R(S \otimes C, K) \to \operatorname{Hom}_R(S, \operatorname{Ext}^1_R(C, M)) \to 0,$$

where we used that the functors $\operatorname{Hom}_R(S, \operatorname{Hom}_R(C, \cdot))$ and $\operatorname{Hom}_R(S \otimes C, \cdot)$ are naturally isomorphic. Since $S \otimes C$ is almost projective, the following sequence is almost exact:

$$0 \longrightarrow \operatorname{Hom}_R(S \otimes C, M) \longrightarrow \operatorname{Hom}_R(S \otimes C, Q) \longrightarrow \operatorname{Hom}_R(S \otimes C, K) \longrightarrow 0,$$

so we have $\mathrm{Hom}_R(S, \mathrm{Ext}^1_R(C, M)) \approx 0$.

Assume by contradiction that $\mathrm{Ext}^1_R(C, M) \not\approx 0$: there is an element $x \in \mathrm{Ext}^1_R(C, M)$ such that $IR \not\subseteq \mathrm{Ann}(x)$. By Lemma 2.3.8, since S is almost faithfully flat, $IR \subseteq \mathcal{E}_{S/R}$: there is an element $a \in \mathcal{E}_{S/R} \setminus \mathrm{Ann}(x)$. Take any $f : S \to R$ such that for some $b \in S$ $f(b) = a$: f induces a map $\tilde{f} : S \to R/\mathrm{Ann}(x) \cong \langle x \rangle_R \subseteq \mathrm{Ext}^1_R(C, M)$ which is not killed by all the elements in I: it follows that $\mathrm{Hom}_R(S, \mathrm{Ext}^1_R(C, M)) \not\approx 0$, which is a contradiction. □

Remark 2.4.17. The hypothesis of being unramified was not used in this proof, since we only needed that S is almost projective and almost faithfully flat as an R-module

Remark 2.4.18. Since this is the transposition to the language of almost mathematics of the property that finite étale coverings split, we can say that almost finite étale coverings almost split.

Remark 2.4.19. If we operate with the additional condition that the base ring is complete, a finite étale covering has a much stronger property: it splits as a map of algebras. However, we will not dive deeper into this result, since it will not be needed in this thesis.

In the following chapter we will introduce some basic concepts of perfectoid theory from Scholze's article [Sch12]. In this context we will finally formulate Faltings' almost purity theorem.

Chapter 3

Perfectoid spaces

3.1 Perfectoid algebras and almost mathematics

Here we will present the basic definitions and results regarding perfectoid fields and algebras - details can be found in [Sch12]. Many of the following propositions will not be directly used in the proof of the main theorem, but they are collected here in an orderly fashion as to provide context for the mathematics used in the next subsection.

Definition 3.1.1. Let's give some definitions for the language that will be used in this chapter.

- Given a field K, we will call *valuation* a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that:
  - we have $|x| = 0$ if and only if $x = 0$;
  - for all $x, y \in K$, $|xy| = |x| \cdot |y|$;
  - for all $x, y \in K$, $|x + y| \leq \max\{|x|, |y|\}$.

  In this case, the couple $(K, |\cdot|)$ is called a *normed field*, and the image of $K \setminus \{0\}$ via $|\cdot|$ is called the *group of valuation*.

- The normed field $(K, |\cdot|)$ is said to be complete if it is complete with respect the topology induced by $|\cdot|$.

- A Banach K-algebra is a couple $(R, ||\cdot||)$, where R is a K algebra and $||\cdot|| : R \to \mathbb{R}_{\geq 0}$ is a function such that:
  - we have $||x|| = 0$ if and only if $x = 0$;
  - for all $x, y \in K$, $||xy|| \leq ||x|| \cdot ||y||$;
  - for all $x, y \in K$, $||x + y|| \leq ||x|| + ||y||$.
  - for all $x \in R$, for all $\lambda \in K$, we have $||\lambda x|| = |\lambda| \cdot ||x||$.
  - the ring R is complete with respect to the topology induced by $||\cdot||$.

- The ring R° of *powerbounded* elements consists of all the elements $x \in R$ such that the set of positive real numbers $\{||x^n||\}_n$ is bounded. In particular, if $R = K$, $K^\circ = \{x \in K | |x| \leq 1\}$ is a local ring.

- The set $R^{\circ\circ}$ of *topologically nilpotent* elements consists of all the elements $x \in R$ such that $\liminf_n ||x^n|| = 0$. In particular, if $R = K$, $K^{\circ\circ} = \{x \in K || x| < 1\}$ is the maximal ideal of K° and it will be called $\mathfrak{m}$.

Example 3.1.2. The ring of p-adic integers $\mathbb{Z}_p$ admits a valuation $|\cdot|_p$ that maps any non zero element x to $p^{-e(x)}$, where $e(x)$ is the maximal exponent such that $p^{e(x)}$ divides x. This valuation can be extended to its fraction field $\mathbb{Q}_p$, the field of p-adic numbers, making $(\mathbb{Q}_p, |\cdot|_p)$ a complete normed field.

Let's define perfectoid fields.

Definition 3.1.3. A perfectoid field is a complete normed field $(K, |\cdot|)$ such that:

- its group of valuation $\Gamma \subseteq \mathbb{R}_{\geq 0}$ is nondiscrete;
- the local ring $(K^\circ, \mathfrak{m})$ has residue characteristic p;
- the Frobenius endomorphism on K°/p (i.e. the map that sends x to x^p) is surjective.

For every perfectoid field we will choose $\omega \in K^\circ$ a fixed nonzero element with $|p| \leq \omega < 1$ (if K has characteristic 0, we can take $\omega = p$).

Remark 3.1.4. If K has characteristic p, $K^\circ/p = K^\circ$, and the Frobenius endomorphism is surjective if and only if K is a perfect field.

Example 3.1.5. Consider the ring $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$ as constructed in Example 2.0.1. We can extend (in a unique way) the p-adic valuation $|\cdot|_p$ to this ring, and complete it with respect to the induced topology. We will denote the resulting ring, its quotient field, and its maximal ideal respectively in the following way:

$$\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}, \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}, (p^{\frac{1}{p^\infty}}).$$

It's easy to check that:

- the field $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ is a perfectoid field;
- the ring $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$ is its local ring of powerbounded elements;
- the ideal $(p^{\frac{1}{p^\infty}})$ is the ideal of topologically nilpotent elements.

From now on, K will always denote a perfectoid field.

Lemma 3.1.6. *The group of valuations Γ is (multiplicatively) p-divisible.*

Proof. For every $\bar{x} \in K^\circ/p \setminus \{0\}$, any lifting $x \in K^\circ$ has norm greater than $|p|$, therefore they all have the same norm. Since the Frobenius is surjective, for every $x \in K^\circ$ with $|p| < |x| \leq 1$, $\bar{x} \neq 0$ and there is $y \in K^\circ$ such that $\bar{y}^p = \bar{x}$, therefore $|y|^p = |x|$. Since the norm is nondiscrete, there is at least an x with $|p| < |x| < 1$, and every element of K° can be recursively written as a product of elements with this property: thus for every $x \in K^\circ$ there is $y \in K^\circ$ such that $|y|^p = |x|$. □

The main concept around which much of the perfectoid theory is built is the *tilting operation*, a construction that we will explain with the following chain of propositions.

Proposition 3.1.7. *Consider the ring K°/ω, of characteristic p, and call $\Phi : x \mapsto x^p$ its Frobenius endomorphism. The projection $\pi : K^\circ \to K^\circ/\omega$ induces an isomorphism of topological multiplicative monoids: $\tilde{\pi} : \varprojlim_{x\to x^p} K^\circ \to \varprojlim_{\Phi} K^\circ/\omega$. In particular, there is a natural multiplicative map $\varprojlim_{x\to x^p} K^\circ \to K^\circ$ which sends f to $f^\#$, given by the projection on the first coordinate.*

Proof. Let's construct a map from $\varprojlim_{\Phi} K^\circ/\omega$ to K. Any element on the left hand side can be thought of as a sequence $x := (\bar{x}^{\frac{1}{p^n}})_n$; take two different lifting of each element in K°: $(x_n)_n$ and $(x'_n)_n$. Let's consider the sequences $x_n^{p^n}$ and $x'^{p^n}_n$ and write $x'_n - x_n = a\omega$, with $a \in K^\circ$; then $x_n^{p^n} - x'^{p^n}_n = bp^n + a^{p^n}\omega^{p^n}$, which is divisible by ω^n: if the two sequences have a limit in K°, it does not depend on the lifting.

For the existence, remember that x_{n+1}^p and x_n are congruent modulo ω: call their difference $c\omega$ with $c \in K^\circ$. Like before, $x_{n+1}^{p^{n+1}} - x_n^{p^n}$ is divisible by ω^n, therefore the sequence converges. Call $x^\#$ this limit: our map will send x to $x^\#$.

Let's prove the multiplicativity: if x lifts to $(x_n)_n$ and y lifts to $(y_n)_n$, one lifting of xy is $(x_n y_n)_n$, and $(xy)^\# = \lim_n (x_n y_n)^{p^n} = \lim_n x_n^{p^n} \lim_n y_n^{p^n} = x^\# y^\#$. For the continuity, if x and y coincide on the first k coordinates, then we can lift them to sequences $(x_n)_n$ and $(y_n)_n$ that also coincide on the first k coordinates. Since ω divides $x_{k+n}^{p^n} - y_{k+n}^{p^n}$, like before $x_{k+n}^{p^{k+n}} - y_{k+n}^{p^{k+n}}$ is divisible by ω^k: the same will be true for $x^\# - y^\#$.

Now, if $x = (\bar{x}^{\frac{1}{p^n}})_n \in \varprojlim_{\Phi} K^\circ/\omega$, we can call $x^{\frac{1}{p^k}} := (\bar{x}^{\frac{1}{p^{k+n}}})_n$, and define a (continuous and multiplicative) map $\tilde{h} : \varprojlim_{\Phi} K^\circ/\omega \to \varprojlim_{x\to x^p} K^\circ$ which sends $x = (\bar{x}^{\frac{1}{p^n}})_n$ to $((x^{\frac{1}{p^n}})^\#)_n$. Since $(x^{\frac{1}{p^n}})^\# \equiv \bar{x}^{\frac{1}{p^n}} \mod \omega$, $\tilde{\pi} \circ \tilde{h}$ is the identity. Conversely, every $(x^{\frac{1}{p^n}})_n \in \varprojlim_{x\to x^p} K^\circ$ is a lift of its own projection, therefore $(\tilde{\pi}(x^{\frac{1}{p^n}})_n)^\# = \lim_n (x^{\frac{1}{p^n}})^{p^n} = x$; it follows that $\tilde{h} \circ \tilde{\pi}$ is the identity, therefore both maps were isomorphisms. □

Remark 3.1.8. Since Φ is a ring homomorphism, $\varprojlim_{\Phi} K^\circ/\omega$ is a ring, therefore the isomorphism we just found induces a natural ring structure on $\varprojlim_{x\to x^p} K^\circ$.

Proposition 3.1.9. *The norm on K induces on $\varprojlim_{\Phi} K^\circ/\omega$ a multiplicative map $f \mapsto |f^\#|$: it is a valuation.*

Proof. To show that the map $f \mapsto |f^\#|$ is a valuation, it suffices to check that, for all the elements $f, g \in \varprojlim_{\Phi} K^\circ/\omega$, $|(f+g)^\#| \leq \max\{|f^\#|, |g^\#|\}$. We will write $f = (\bar{f}^{\frac{1}{p^n}})_n$ and $g = (\bar{g}^{\frac{1}{p^n}})_n$. From the proof of the previous proposition, we know that for some big enough m, any lifting $\tilde{f}$ of $\bar{f}^{\frac{1}{p^m}}$ is such that $|\tilde{f}^{p^m}| = |f^\#|$; moreover, we can take an m such that this is true at the same time for f, g, and $f+g$. If we choose the liftings $\tilde{f}$, $\tilde{g}$, and $\tilde{fg}$ respectively of $\bar{f}^{\frac{1}{p^m}}$, $\bar{g}^{\frac{1}{p^m}}$, and $\bar{fg}^{\frac{1}{p^m}}$, such that $\tilde{f} + \tilde{g} = \tilde{fg}$, we get that:

$$|(f+g)^\#| = |(\tilde{f} + \tilde{g})|^{p^m} \leq \max\{|\tilde{f}|, |\tilde{g}|\}^{p^n} = \max\{|\tilde{f}|^{p^n}, |\tilde{g}|^{p^n}\} = \max\{|f^\#|, |g^\#|\}.$$

□

Proposition 3.1.10. *There is an element $\omega^\flat \in \varprojlim_{\Phi} K^\circ/\omega$ such that $|(\omega^\flat)^\#| = |\omega|$.*

Proof. Take $x \in K^\circ$ such that $|x|^p = |\omega|$, and let $\bar{x} \in K^\circ/\omega \setminus \{0\}$ be its projection. Consider a system of p power roots $\tilde{x} := (\bar{x}^{\frac{1}{p^n}})_n$ (which exists because the Frobenius is surjective), and take its p-th power: $\omega^\flat := (0, \bar{x}, \bar{x}^{\frac{1}{p}}, \cdots)$. As observed in the previous propositions, $(\tilde{x})^\# \equiv x \mod \omega$, therefore $|(\tilde{x})^\#| = |x|$, which implies that $|(\omega^\flat)^\#| = |(\tilde{x}^p)^\#| = |(\tilde{x})^\#|^p = |x|^p = |\omega|$. □

Definition 3.1.11. We define K's tilt as $K^\flat := \varprojlim_\Phi K^\circ/\omega[(\omega^\flat)^{-1}]$.

Remark 3.1.12. The multiplicative map $f \to f^\#$ constructed in Proposition 3.1.7 can be extended to $K^\flat \to K$, and $K^\flat$ also inherits the norm of Proposition 3.1.9.

Without losses of generality, from here on out we will take $\omega := (\omega^\flat)^\#$; in particular, ω will have a system of p power roots, given by $\left((\omega^\flat)^{\frac{1}{p^n}}\right)^\#$.

Remark 3.1.13. With this assumption, since $\mathfrak{m} \subseteq K^\circ$ consists of all the elements with norm strictly less than 1, we can think of $\{\omega^{\frac{1}{p^k}}\}_k$ as a set of generators for $\mathfrak{m}$. Since this is the same requirement as Chapter 2, we may talk about almost mathematics (with respect to ω) also in the context of perfectoid theory.

Proposition 3.1.14. *The natural norm $K^\flat \ni f^\flat \mapsto |(f^\flat)^\#|$ induces the limit topology on $\varprojlim_\Phi K^\circ/\omega$. Moreover, $K^{\flat\circ}$ is exactly $\varprojlim_\Phi K^\circ/\omega$, and $K^\flat$ is a perfectoid field.*

Proof. Let's prove the second part of the statement first.

Obviously, $\varprojlim_{x\to x^p} K^\circ \subseteq K^{\flat\circ}$ by definition of the norm on $K^\flat$. Vice versa, take $f^\flat \in K^\flat$ with $|(f^\flat)^\#| \leq 1$. By definition of $K^\flat$ there is some N such that $(\omega^\flat)^N f^\flat \in \varprojlim_{x\to x^p} K^\circ$; write $(\omega^\flat)^N f^\flat = (f_0, f_1, \cdots)$. Denoting with $\omega^{\frac{1}{p^n}}$ the image in K of the only p^n-th root of $\omega^\flat \in K^\flat$, we have:

$$\left(\frac{f_n}{\omega^{\frac{N}{p^n}}}\right)^{p^n} = \frac{f_0}{\omega^N} = \frac{\left((\omega^\flat)^N f^\flat\right)^\#}{\left((\omega^\flat)^N\right)^\#} = (f^\flat)^\# \in K^\circ.$$

So we can take the sequence $(f_n\omega^{-\frac{N}{p^n}})_n \in \varprojlim_{x\to x^p} K^\circ$. If multiplied by $(\omega^\flat)^N = (\omega^{-\frac{N}{p^n}})_n$, it yields $(\omega^\flat)^N f^\flat$, therefore $f^\flat = (f_n\omega^{-\frac{N}{p^n}})_n \in \varprojlim_{x\to x^p} K^\circ$.

Now we want to prove that the Frobenius map on ${K^\flat}^\circ$ is surjective, that the norm on $K^\flat$ is nondiscrete, and that $K^\flat$ is complete with respect to the topology induced by the norm.

The surjectivity of the Frobenius map follows from the identification of ${K^\flat}^\circ$ we just gave. This also implies that the norm is nondiscrete, since it suffices to find elements with norm arbitrarily close to 1, and to do that we can consider all the p-power roots of $\omega^\flat$.

To show that $(K^\flat, |\cdot^\#|)$ is complete, it suffices to prove it for ${K^\flat}^\circ$, and we can think its elements as successions in $\varprojlim_\Phi K^\circ/\omega$. Consider a Cauchy sequence with respect to the norm $|\cdot^\#|$: $\{(x_n^{(0)})_n, (x_n^{(1)})_n, (x_n^{(2)})_n, \cdots\}$. For all k, there is m such that for all $i, j > m$, $(x_n^{(i)} - x_n^{(j)})_n$ has norm less that $|\omega|^{p^k}$; in particular, $x_n^{(i)} - x_n^{(j)} = 0$ for all $n \leq k$. Therefore, we get that these sequences converge pointwise (i.e. in the limit topology of $\varprojlim_\Phi K^\circ/\omega$) to some element. With the completeness, we have proven that $(K^\flat, |\cdot^\#|)$ is a perfectoid field.

For the first part of the statement, note that we have just shown that Cauchy sequences in the topology induced by the norm are the same as converging sequences with respect to the limit topology, and the inverse is obvious by a similar reasoning.

□

Proposition 3.1.15. *We have the following. $K^\flat$ and $\varprojlim_{x\to x^p} K$ are isomorphic as topological multiplicative monoids. Moreover, if $K = p$, $K^\flat = K$.*

Proof. Fix $x \in K$ and take $(\bar{x}^{\frac{1}{p^n}})_n \in \varprojlim_{x\to x^p} K$. The element x can be written as $\frac{y}{\omega^k}$ for some k, where $\omega := (\omega^\flat)^\#$, and $y \in K^\circ$. If $\omega^\flat = (\bar{\omega}^{\frac{1}{p^n}})_n$ (with $n > 0$), we can write:

$$(\bar{x}^{\frac{1}{p^n}})_n = \left((\bar{y}(\bar{\omega})^{-k})^{\frac{1}{p^n}}\right)_n = (\bar{y}^{\frac{1}{p^n}})_n(\omega^\flat)^{-k},$$

so $\varprojlim_{x \to x^p} K \subseteq K^\flat$. Vice versa, as we already wrote, $(\omega^\flat)^{-1} = \left(\frac{1}{\bar{\omega}^{\frac{1}{p^n}}}\right)_n$, so the other containment is also true.

Finally, if $\mathrm{char} K = p$, $\varprojlim_{x \to x^p} K \cong K$ via the map that sends $(x^{\frac{1}{p^n}})_n$ to x, because every element $x \in K$ admits a system of p-power roots. □

Example 3.1.16. Let's consider again $K := \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$, with $K^\circ = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$, and let's compute its tilt. We can take $\omega = p$, therefore:

$$\varprojlim_{\Phi} K^\circ/\omega = \varprojlim_{\Phi} \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}/p \cong \varprojlim_{t \mapsto t^p} \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t \cong \mathbb{F}_p[[t]][t^{\frac{1}{p^\infty}}].$$

Thus $K^\flat \cong \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, and in this case we can take $\omega^\flat = t$.

After this preliminary work, we may present the main classes of objects studied in perfectoid theory (although we will mainly need just the first category from the following definition).

Definition 3.1.17. As always, let K be a perfectoid field.

- A perfectoid K-algebra R is a Banach K-algebra such that the set of powerbounded elements $R^\circ \subset R$ is open and bounded, and the Frobenius endomorphism on R°/ω is surjective. A morphism of perfectoid K-algebras is a morphism of Banach K-algebras, and the corresponding category is called $K - perf$.
- A perfectoid $K^{\circ a}$-algebra A is an almost ω-adically complete and almost flat $K^{\circ a}$-algebra A on which the Frobenius endomorphism induces an almost isomorphism $A/\omega \cong A/\omega^{\frac{1}{p}}$. A morphism of perfectoid $K^{\circ a}$-algebras is simply a morphism of $K^{\circ a}$-algebras, and the corresponding category is called $K^{\circ a} - perf$.
- A perfectoid $K^{\circ a}/\omega$-algebra is an almost flat $K^{\circ a}/\omega$-algebra $\bar{A}$ on which the Frobenius endomorphism induces an almost isomorphism $\bar{A} \cong \bar{A}/\omega^{\frac{1}{p}}$. A morphism of perfectoid $K^{\circ a}/\omega$-algebras is simply a morphism of $K^{\circ a}/\omega$-algebras, and the corresponding category is $K^{\circ a}/\omega - perf$.

Example 3.1.18. The main example of a perfectoid K-algebra is $R = K\langle T^{\frac{1}{p^\infty}}\rangle$, which is the algebra obtained by adjoining a variable T and all its p-power roots to K, and taking the ω-adic completion (we can think of this object as the analogous in perfectoid theory to the ring $\mathbb{C}[t]$ in complex geometry).

To better understand the meaning of a perfectoid $K^{\circ a}$-algebra, let's explore the definition with the following proposition.

Proposition 3.1.19. *Take $M \in K^{\circ a} - \mathbf{Mod}$.*

1. *The following are equivalent:*
   - *as a $K^{\circ a}$-module M is almost flat;*
   - *as a K°-module, M_* is flat;*
   - *the K°-module M_* has no ω-torsion.*
2. *If $N \in K^\circ - \mathbf{Mod}$ is flat and $M = N^a$, then M is almost flat and $M_* \cong \tilde{N}$, where $\tilde{N} = \{x \in N[\omega^{-1}] | x\mathfrak{m} \subseteq N\}$.*

3. *If M is almost flat, then $(xM)_* = x(M_*)$ for all $x \in K^\circ$. Moreover $M_*/xM_* \subseteq (M/xM)_*$ contains the image of the natural map $(M/x\omega^\alpha M)_* \to (M/xM)_*$ for all $\alpha > 0$.*

4. *If M is almost flat, then M is almost ω-adically complete (i.e. it is almost isomorphic to $\varprojlim_n M/\omega^n M$, in the category $K^{\circ a} - \mathbf{Mod}$) if and only if M_* is ω-adically complete.*

Proof. 1. By Proposition 2.1.7 we know that $(M_*)^a \approx M$, therefore by definition M is almost flat if $\mathrm{Tor}^1_{K^\circ}(M_*, N) \approx 0$ for every K°-module N, which is implied by M_*'s flatness.

Suppose M to be almost flat. Tensoring by M_* the inclusion $\omega K^\circ \subseteq K^\circ$ we get an almost injective map $M_* \to M_*$, whose kernel N is comprised of the elements of ω-torsion in M_*. Always by Proposition 2.1.7 we know that $M_* \cong (M_*^a)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*)$, and the isomorphism sends $x \in M_*$ to the map $a \to ax$: in particular, for all $x \in M_*$ there is an $a \in \mathfrak{m}$ such that $ax \neq 0$, which means that there are no almost zero elements in M_*. As a result, N must be 0, so there are no ω-torsion elements in M_*.

Suppose now that M_* has no ω-torsion, which means that for every k it has no $\omega^{\frac{1}{p^k}}$-torsion. An equivalent condition for flatness is that $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I) = 0$ for every proper ideal $I \subsetneq K^\circ$, and these can be of only two types: $I_\alpha := (\omega^\alpha)$ and $J_\alpha := \bigcup_{\beta > \alpha} I_\beta$.

Consider the exact sequence $0 \longrightarrow K^\circ \xrightarrow{\cdot\omega^\alpha} K^\circ \longrightarrow K^\circ/I_\alpha \longrightarrow 0$. Tensoring by M_* we get:

$$0 \longrightarrow \mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I_\alpha) \longrightarrow M_* \xrightarrow{-\cdot\omega^\alpha} M_*,$$

but since M_* has no ω^α-torsion, $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I_\alpha) = 0$. For J_α, call j its map of inclusion into K°; we get the following commutative diagrams for every $\beta > \alpha$:

$$\begin{array}{ccccc} 0 & \longrightarrow & K^\circ & \xrightarrow{-\cdot\omega^\beta} & K^\circ \\ \downarrow & & \downarrow{\scriptstyle -\cdot\omega^\beta} & & \| \\ 0 & \longrightarrow & J_\alpha & \xrightarrow{j} & K^\circ \end{array} \quad \Rightarrow \quad \begin{array}{ccccc} 0 & \longrightarrow & M_* & \xrightarrow{-\cdot\omega^\beta} & M_* \\ \downarrow & & \downarrow{\scriptstyle i_\beta} & & \| \\ 0 & \longrightarrow \mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/J_\alpha) \longrightarrow & M_* \otimes J_\alpha & \xrightarrow{id_{M_*} \otimes j} & M_*. \end{array}$$

Tensoring by M_* commutes with colimits, and J_α is the colimit of a diagram $\{K^\circ\}_{\beta > \alpha}$ whose maps towards the colimit are $-\cdot\omega^\beta$. Therefore, we get that $M_* \otimes J_\alpha$ is the colimit of a diagram $\{M_*\}_{\beta > \alpha}$ whose maps towards the colimit are i_β. In particular, every element $y \in \ker(id_{M_*} \otimes j$ can be written as $i_\beta(x)$ for some $\beta > \alpha$ and some $x \in M_*$, so we have:

$$0 = id_{M_*} \otimes j(y) = (id_{M_*} \otimes j) \circ i_\beta(x) = x\omega^\beta,$$

which means that $x = 0$ because M_* has no ω^β-torsion. This means that $id_{M_*} \otimes j_\alpha$ is injective and $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/J_\alpha) = 0$.

2. Again, N flat implies $M = N^a$ almost flat. Consider the morphism $\phi : \tilde{N} \to M_*$ that sends x to the map $a \mapsto ax$ in $\mathrm{Hom}_{K^\circ}(\mathfrak{m}, N) \cong M_*$. Let's prove ϕ is injective. Take $y = \frac{x}{\omega^n} \in \ker(\phi)$, with $x \in N$. Since $\phi(y) = 0$, we have $y\omega^n = 0$, therefore $x = 0$. For the surjectivity, take $f \in \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N)$ and call $x := f(\omega)$. For any $a \in \mathfrak{m}$, multiplication by $\frac{x}{\omega}$ yields:

$$\frac{ax}{\omega} = \frac{af(\omega)}{\omega} = \frac{f(a\omega)}{\omega} = \frac{\omega f(a)}{\omega} = f(a) \in N,$$

therefore $\frac{x}{\omega} \in \tilde{N}$, and its image via ϕ is f.

3. Since $(M_*)^a \approx M$, $(x(M_*))^a \approx xM$, but $x(M_*)$ has no ω-torsion, therefore it is flat and by the previous point $x(M_*) = (xM)_*$. Since $(-)_*$ is right adjoint to $(-)^a$, it is left exact, so the exact sequence $0 \to xM \to M \to M/xM \to 0$ yields the following exact sequence: $0 \to (xM)_* \to M_* \to (M/xM)_*$; in particular there is an inclusion $M_*/(xM)_* \hookrightarrow (M/xM)_*$. Moreover, the respective almost modules are the same because the composition of functors $((-)_*)^a$ is isomorphic to the identity on $K^{\circ a} - \mathbf{Mod}$, therefore:

$$(M/xM)_* = (((M/xM)_*)^a)_* = ((M_*/(xM)_*)^a)_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/(xM)_*).$$

Consider the natural map $(M/x\omega^\alpha M)_* \to (M/xM)_*$ induced by multiplication by ω^α. If $m \in (M/xM)_*$ can be lifted to $\tilde{m} \in (M/x\omega^\alpha M)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/x\omega^\alpha M_*)$, call $n := \tilde{m}(\omega^\alpha)$ and lift it to $\tilde{n} \in M_*$. The element $\tilde{n}$ is divisible by ω^α if $\omega^\alpha | a\tilde{n}$ for every $a \in \mathfrak{m}$, and we have:

$$\overline{a\tilde{n}} = an = a\tilde{m}(\omega^\alpha) = \tilde{m}(a\omega^\alpha) = \omega^\alpha \tilde{m}(a).$$

Now let's take $n' := \frac{\tilde{n}}{\omega^\alpha} \in M_*$. The natural map $M_* \to (M/xM)_*$ sends n' to the following map in $\mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/xM_*)$:

$$a \mapsto \overline{an'} = a\overline{n'} = \frac{an}{\omega^\alpha} = \frac{a\tilde{m}(\omega^\alpha)}{\omega^\alpha} = \frac{\tilde{m}(a\omega^\alpha)}{\omega^\alpha} = \frac{\omega^\alpha \tilde{m}(a)}{\omega^\alpha} = \tilde{m}(a).$$

4. As already seen in 2.1.7, the functor $(-)_*$ is right adjoint to the functor $(-)^a$, and $(-)^a$ is right adjoint to the functor $(-)_!$, so they commute with limits. If M is almost ω-adically complete, we have:

$$M_* = (\varprojlim_n M/\omega^n M)_* = \varprojlim_n (M/\omega^n M)_* \cong \varprojlim_n M_*/\omega^n M_*,$$

where the last isomorphism is due to the fact that $M_*/\omega^n M_* \subseteq (M/\omega^n M)_*$ and the transition map $(M/\omega^{n+1}M)_* \to (M/\omega^n M)_*$ has image contained in $M_*/\omega^n M_*$ for all n, as proven in the previous point.

Vice versa, if M_* is ω-adically complete:

$$M \approx (M_*)^a = (\varprojlim_n M_*/\omega^n M_*)^a = \varprojlim_n (M_*/\omega^n M_*)^a \approx \varprojlim_n M/\omega^n M,$$

where the last almost isomorphism is a direct consequence of point (3).

□

Remark 3.1.20. Given a perfectoid K-algebra, one can define its tilt in the same way as for K, to obtain a perfectoid $K^\flat$-algebra. A fundamental result of perfectoid theory states that the tilting functor induces an equivalence of the corresponding categories of perfectoid algebras $K - perf \cong K^\flat - perf$ ([Sch12, Theorem 5.2]); this has a nice philosophical implication: that studying rings of mixed characteristic $(0, p)$ is equivalent to studying rings of characteristic p if one works in the context of perfectoid algebras.

Finally, we may formulate in the language of perfectoid algebras Faltings' almost purity theorem:

Theorem 3.1.21 ([Sch12, Theorem 7.9.iii])**.** *Let R be a perfectoid K-algebra, and $R \to S$ a finite étale extension. Then S is perfectoid and S° is almost finite étale over R°.*

3.2 Perfectoid spaces

This section follows [Sch12, Chapter 6]. Scholze's aim is to assign a topological space and an associated sheaf to a given perfectoid K-algebra, in the same way as with adic spaces.

We will start with some basic construction, and talk about the results we need, mainly the lemmas 3.2.6, 3.2.7, and 3.2.8. After that, for the sake of completeness, we will highlight the main results of [Sch12, Chapter 6], of which the aforementioned lemmas are very particular cases: Lemma 3.2.9 and Theorem 3.2.10.

First, we construct the underlying space.

Definition 3.2.1. A perfectoid affinoid K-algebra is a pair (R, R^+), where R is a perfectoid K-algebra and $R^+ \subseteq R^\circ$ is an open and normal subring.

$$X := Spa(R, R^+) = \{v : R \to \Gamma \cup \{0\} \text{ continuous valuation} | \forall f \in R^+ : v(f) \leq 1\} / \sim,$$

where Γ is the value group of K. For any $x \in X$ we will write the associated valuation as $f \mapsto |f(x)|$. This space will be given the topology generated by the following so-called rational subsets:

$$U\left(\frac{f_1 \cdots f_n}{g}\right) := \{x | \forall i \ |f_i(x)| \leq |g(x)|\},$$

where $g \in R$ and $f_1, \cdots, f_n \in R$ generate R as an R-module.

Remark 3.2.2. Since R^+ is integrally closed in R°, it can be easily shown that $\mathfrak{m}R^\circ \subseteq R^+ \subseteq R^\circ$. In particular, all the possible choices for R^+ are almost isomorphic to R°.

We now associate a pair of rings to all rational subsets.

Definition 3.2.3. Let $U = U\left(\frac{f_1, \cdots, f_n}{g}\right)$. Choose an open subring $R_0 \subseteq R$ such that $\{aR_0\}_{a \in K^*}$ is a basis of open neighborhoods of 0. Consider the algebra $R[g^{-1}]$ and equip it with the topology making $\{aR_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]\}_{a \in K^*}$ a basis of open neighborhoods of 0. Finally let $B \subseteq R[g^{-1}]$ the normalization of $R^+[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]$. We define the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ as the completion of the pair $(R[\frac{f_1}{g}, \cdots, \frac{f_n}{g}], B)$.

Remark 3.2.4. Scholze proves that this association can be extended to all open subsets by taking a limit, making $\mathcal{O}_X$ and $\mathcal{O}_X^+$ two sheaves on X such that, for any open set W, the pair $(\mathcal{O}_X(W), \mathcal{O}_X^+(W))$ will itself be a perfectoid affinoid K-algebra. Anyway, we won't need this sort of construction in this thesis.

Remark 3.2.5. We can define a natural map $X \to X^\flat$ that sends the valuation $R \ni f \mapsto |f(x)|$ to the valuation $R^\flat \ni g \mapsto |g^\#(x)|$. The preimage of a generic rational subset $U\left(\frac{f_1, \cdots, f_n}{g}\right) \subseteq X^\flat$ is given by the rational subset $U\left(\frac{f_1^\#, \cdots, f_n^\#}{g^\#}\right) \subseteq X$, therefore this map is continuous.

We present the following approximation lemma, the quite technical proof of which we won't include.

Lemma 3.2.6 ([Sch12, Corollary 6.7.i])**.** *For any $f \in R$ and any $c > 0, \varepsilon > 0$, there is $h(f)_{c,\varepsilon} \in R^\flat$ such that for all $x \in X$ we have $|f(x) - h(f)^\#_{c,\varepsilon}(x)| \leq |\omega|^{1-\varepsilon} \max\{|f(x)|, |\omega|^c\}$.*

With the help of Lemma 3.2.6, it's possible to show that every rational subset in X can be written as a preimage of some rational subset in $X^\flat$ via the map defined in Remark 3.2.5. We will only prove this result in the following particular case.

Proposition 3.2.7. *Let $U \subseteq X$ be a rational subset of the form $U := U\left(\frac{f,\omega^n}{\omega^n}\right)$ for some $f \in R^\circ$, $n > 0$. Then, there is an element $f^\flat \in (R^\flat)^\circ$ such that:*

- *the rational subsets U and $U\left(\frac{(f^\flat)^\#,\omega^n}{\omega^n}\right)$ coincide;*
- *we have the congruence $(f^\flat)^\# \equiv f \mod \omega^{\frac{1}{p}}$.*

Proof. If we take $f^\flat := h(f)_{n,1-\frac{1}{p}}$, defined as in Lemma 3.2.6, to prove that $U\left(\frac{f,\omega^n}{\omega^n}\right)$ and $U\left(\frac{(f^\flat)^\#,\omega^n}{\omega^n}\right)$ coincide, we just need to show that there are no $x \in X$ for which only one between $|f(x)|$ and $|(f^\flat)^\#(x)|$ is greater than $|\omega^n|$. If this happened, in particular the two norms would be different, and we would have, by definition of $f^\flat$:

$$\max\{|f(x)|, |(f^\flat)^\#(x)|\} = |f(x) - (f^\flat)^\#(x)| \leq |\omega|^{\frac{1}{p}} \max\{|f(x)|, |\omega|^n\}.$$

If $|f(x)| > |\omega^n| \geq |(f^\flat)^\#(x)|$, we would get that $|f(x)| \leq |\omega|^{\frac{1}{p}}|f(x)|$ which is a contradiction, while if $|(f^\flat)^\#(x)| > |\omega^n| \geq |f(x)|$ we would get that $|(f^\flat)^\#(x)| \leq |\omega|^{\frac{1}{p}}|\omega|^n$, which is again a contradiction.

For the second point, since $f \in R^\flat$, we get that for all $x \in X$ $1 \geq |f(x)| = |(f^\flat)^\#(x)|$. In particular, $f^\flat \in (R^\flat)^\circ$ and:

$$|f(x) - (f^\flat)^\#(x)| \leq |\omega|^{\frac{1}{p}} \max\{|f(x)|, |\omega|^n\} \leq |\omega|^{\frac{1}{p}},$$

which proves the statement. □

Let's prove the following lemma, which allows us to work with a concrete definition of $\mathcal{O}_X^+(U)$ for a particular $U = U\left(\frac{\omega^m}{g}\right)$.

Lemma 3.2.8. *Let $g \in R^\circ$ be an element such that:*

- *the element g is not a zero divisor;*
- *the ring R° contains a system of p-power roots of g;*
- *there is no k such that $\omega^{\frac{1}{p^k}}|g$.*

Fix some m and consider the map:

$$\pi : R^\circ\left[T_1^{\frac{1}{p^\infty}}\right] \to R^\circ\left[\left(\frac{\omega^m}{g}\right)^{\frac{1}{p^\infty}}\right]$$
$$T^{\frac{1}{p^k}} \mapsto \left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}.$$

It is surjective, with kernel:

$$I := ((Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}})_k.$$

Proof. Surjectivity is obvious. Fix some k and consider the induced map:

$$\pi : R^\circ\left[T_1^{\frac{1}{p^k}}\right] \to R^\circ\left[\left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}\right]$$
$$T^{\frac{1}{p^k}} \mapsto \left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}.$$

If we prove that this map has kernel generated by $x_k := (Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}}$, passing to the colimit we get the statement of the lemma.

Inverting ω the right hand side becomes $R^\circ[g^{-1}]$, and the kernel becomes exactly:

$$\left((Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}}\right) R^\circ\left[\omega^{-1}, T_1^{\frac{1}{p^k}}\right],$$

therefore we just have to prove that the intersection between this module and $R^\circ[T_1^{\frac{1}{p^k}}]$ is (x_k). By contradiction, if this were not the case we could find some $f \in R^\circ[T_1^{\frac{1}{p^k}}]$ and $\alpha > 0$ such that $x_k\omega^{-\alpha}f$ is contained in $R^\circ[T_1^{\frac{1}{p^k}}]$, while $\omega^{-\alpha}f$ is not. This implies that there is some $\beta > 0$ such that $\omega^\beta | x_k$, which means that some power of ω divides $(Tg)^{\frac{1}{p^k}}$, which is contrary to the third hypothesis. □

To put this result in context, we present without proof a larger lemma, of which it is a particular case, that explicitly describes $\mathcal{O}_X(U^\#)$ for any rational set $U \subseteq X^\flat$ at the almost-integral level.

Lemma 3.2.9 ([Sch12, Lemma 6.4])**.** *Let $U := U\left(\frac{f_1,\cdots,f_n}{g}\right)$ and $U^\# := U\left(\frac{f_1^\#,\cdots,f_n^\#}{g^\#}\right)$ be rational subsets of $R^\flat$ (with the same notation as Definition 3.2.1).*

(1) Consider the map:

$$R^\circ\left[T_1^{\frac{1}{p^\infty}}, \cdots, T_n^{\frac{1}{p^\infty}}\right] \to R^\circ\left[\left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}}\right]$$
$$T_i^{\frac{1}{p^k}} \mapsto f_i^{\frac{1}{p^k}}.$$

It is surjective, and its kernel is almost isomorphic to the ideal:

$$I := \left(g^{\#\frac{1}{p^k}} T_i^{\frac{1}{p^k}} - f_i^{\#\frac{1}{p^k}}\right)_{i,k}$$

(2) The ring $R^\circ\left\langle\left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}}\right\rangle$ is a perfectoid $K^{\circ a}$-algebra.

(3) $\mathcal{O}_X(U^\#)^\circ (\approx \mathcal{O}_X^+(U^\#))$ is almost isomorphic to $R^\circ\left\langle\left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}}\right\rangle$.

(4) The tilt of $\mathcal{O}_X(U^\#)$ is given by $\mathcal{O}_{X^\flat}(U)$.

Finally, let's state the main theorem of [Sch12, Chapter 6], which makes fully explicit the interplay between the pair $(\mathcal{O}_X, \mathcal{O}_X^+)$ and the tilting operation.

Theorem 3.2.10 ([Sch12, Theorem 6.3])**.** *Let (R, R^+) be a perfectoid affinoid K-algebra, and let $X = Spa(R, R^+)$ with associated presheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$. Also, let $(R^\flat, {R^\flat}^+)$ be its tilt, and let $X^\flat = Spa(R^\flat, {R^\flat}^+)$. Then:*

- *the map $X \to X^\flat$ that sends the valuation $R \ni f \mapsto |f(x)|$ to $R^\flat \ni g \mapsto |g^\#(x)|$ is a homeomorphism;*
- *for every rational subset $U \subseteq X$ with image $U^\flat \subseteq X^\flat$ the pair $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ is a perfectoid affinoid K-algebra, with tilt $\big(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\big)$;*
- *the presheaf $\mathcal{O}_X$ is a sheaf, and $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) || f(x)| \leq 1 \forall x \in U\}$.*

Chapter 4

Almost-pro-modules

In this chapter we talk about pro-modules and their almost analogue, following [Bha18, Section 3-4]. To keep the exposition clean, we will take for granted some notions of category theory.

4.1 Definition of pro-modules

In this section we will give a very quick presentation of pro-modules. Let's first define what we mean by pro-module:

Definition 4.1.1. Let R be a (commutative) ring and a N the poset of natural numbers, viewed as a category where the arrow goes from the bigger number to the smaller. Let $F : \mathbb{N} \to R-\mathbf{Mod}$ be a functor, and compose it with the Yoneda embedding $Y : R - \mathbf{Mod} \to [R - \mathbf{Mod}^{\mathrm{op}}, \mathbf{Set}]$. A pro-$R$-module is the limit of $Y \circ F$ in $[R - \mathbf{Mod}^{\mathrm{op}}, \mathbf{Set}]$.

We can identify a pro-R-module with a sequence:

$$M_0 \xleftarrow[f_0^1]{} M_1 \xleftarrow[f_1^2]{} M_2 \xleftarrow[f_2^3]{} M_3 \xleftarrow[f_3^4]{} \cdots.$$

Given two pro-R-modules, $\{M_i\}_i$ and $\{N_j\}_j$, the morphisms between them are given by:

$$\varprojlim_j \varinjlim_i \mathrm{Hom}_R(M_i, N_j).$$

Concretely, we can think of it as a collection of morphisms $\phi_j : M_{k_j} \to N_j$, where $k_j \geq j$, with the obvious commutation conditions between the ϕ_j's and the transition maps (furthermore, one can always tweak the indexes without changing the isomorphism class so that $i_j = j$).

Remark 4.1.2. The morphism $\{\phi_j\}_j$ is zero if and only if for all j ϕ_j induces the zero map on $M_i \to N_j$ for some large enough i. Consequently, a pro-R-module $\{M_i\}_i$ is zero when its identity is the zero map, i.e. when for all j there is $k \geq j$ such that the transition map $f_j^k : M_k \to M_j$ is zero.

Remark 4.1.3. We can think of an isomorphism between $\{M_i\}_i$ and $\{N_i\}_i$ - after reindexing - as a collection $\{f_i : M_i \to N_i\}_i$ such that $\{\ker(f_i)\}_i$ and $\{\mathrm{coker}(f_i)\}_i$ (with the obvious transition maps) are zero as pro-R-modules.

4.2 Definition of almost-pro-modules

We need to define an analogue version of pro-modules to be used in almost mathematics (with respect to a certain system $\{t^{\frac{1}{p^k}}\}_k$ in R). As objects, almost-pro-R-modules will be the same as pro-R-modules. As usual, the crux of the matter is understanding when to define an object as almost zero - all the other definitions can be obtained in a similar way to what we did in Chapter 2.

Definition 4.2.1. A pro-R-module $\{M_n\}_n$ is said to be *almost-pro-zero* when for all i, k there is $j \geq i$ such that the transition map $f_i^j : M_j \to M_i$, multiplied by $t^{\frac{1}{p^k}}$, is zero.

It is said to be *uniformly almost-pro-zero* if for all k there is some $c > 0$ such that for all i the transition map $f_i^{i+c} : M_{i+c} \to M_i$, multiplied by $t^{\frac{1}{p^k}}$, is zero.

Definition 4.2.2. Let $\{M_n\}_n$, $\{N_n\}_n$ be two pro-R-modules and let $\{f_n : M_n \to N_n\}_n$ be a morphism of pro-R-modules. We say that $\{f_n\}_n$ is an *almost-pro-isomorphism* if the pro-R-modules $\{\ker(f_n)\}_n$ and $\{\operatorname{coker}(f_n)\}_n$ are almost-pro-zero.

Remark 4.2.3. If we denote by $M_n[t^{\frac{1}{p^k}}] \subseteq M$ the submodule of $t^{\frac{1}{p^k}}$-torsion, $\{M_n\}_n$ is almost-pro-zero if and only if, for all k, the natural map $\{M_n[t^{\frac{1}{p^k}}]\}_n \to \{M_n\}_n$ is an isomorphism of pro-R-modules.

To showcase the good behaviour of Definition 4.2.1 let's prove some propositions.

Lemma 4.2.4. *Let the following be a pointwise exact sequence of pro-R-modules:*

$$0 \longrightarrow \{K_n\}_n \xrightarrow{\{f_n\}_n} \{M_n\}_n \xrightarrow{\{g_n\}_n} \{C_n\}_n \longrightarrow 0,$$

where $\{f_n\}_n$ and $\{g_n\}_n$ are morphism of pro-R-modules. If $\{M_n\}_n$ is almost-pro-zero, so are $\{K_n\}_n$ and $\{C_n\}_n$.

Moreover, if $\{M_n\}_n$ is uniformly almost-pro-zero, so are $\{K_n\}_n$ and $\{C_n\}_n$.

Proof. We will prove only the first statement, since the proof for the second is practically the same. Let $\{p_i^j\}_{i,j}, \{q_i^j\}_{i,j}, \{r_i^j\}_{i,j}$ be the transition maps respectively of $\{K_n\}_n, \{M_n\}_n, \{C_n\}_n$. For all i, for all k, there is some $j \geq i$ such that the map $t^{\frac{1}{p^k}} q_i^j$ is identically zero. We have:

$$0 = t^{\frac{1}{p^k}} q_i^j \circ f_j = t^{\frac{1}{p^k}} f_i \circ p_i^j,$$

and since f_i is injective, $t^{\frac{1}{p^k}} p_i^j = 0$. Similarly, we have:

$$0 = t^{\frac{1}{p^k}} g_i \circ q_i^j = t^{\frac{1}{p^k}} r_i^j \circ g_j,$$

and since g_j is surjective, $t^{\frac{1}{p^k}} r_i^j = 0$. □

Proposition 4.2.5. *Let $\{M_n\}_n$ be an almost-pro-zero pro-R-module. Then $\varprojlim_n M_n$ is almost zero.*

Proof. Call $\{p_i^j\}$ the transition maps of $\{M_n\}_n$. Fix a sequence $(x_n)_n$ which identifies an element in $\varprojlim_n M_n$. For all k and for all i, there is some $j \geq i$ such that the map $t^{\frac{1}{p^k}} p_i^j$ is identically zero, therefore $0 = t^{\frac{1}{p^k}} p_i^j(x_j) = t^{\frac{1}{p^k}} x_i$. This means that for all k $t^{\frac{1}{p^k}} (x_n)_n = 0$, which implies that $\varprojlim_n M_n$ is almost zero. □

Proposition 4.2.6. *Let $\{M_n\}_n$ be an almost-pro-zero pro-R-module, and consider an R-linear functor $F : R - \mathbf{Mod} \to R - \mathbf{Mod}$. Then $\{F(M_n)\}_n$ is an almost-pro-zero pro-R-module.*

Moreover, if $\{M_n\}_n$ is uniformly almost-pro-zero, so is $\{F(M_n)\}_n$.

Proof. Like before, we will prove only the first statement, since the proof for the second is practically the same. Fix two integers $n, k > 0$. Choose m such that the $f_n^m : M_m \to M_n$ is $t^{\frac{1}{p^k}}$-torsion, i.e. $f_n^m(M_m) \subseteq M_n[t^{\frac{1}{p^k}}]$; applying F we get that $Ff_n^m : F(M_m) \to F(M_n)$ factors through $F(M_n[t^{\frac{1}{p^k}}])$. Since F is R-linear, it preserves the endomorphism $- \cdot t^{\frac{1}{p^k}}$; on the module $M_n[t^{\frac{1}{p^k}}]$ this endomorphism is zero, therefore the same happens for $F(M_n[t^{\frac{1}{p^k}}])$, and we get a natural map $F(M_n[t^{\frac{1}{p^k}}]) \to F(M_n)[t^{\frac{1}{p^k}}]$. This means that Ff_n^m factors through $F(M_n)[t^{\frac{1}{p^k}}]$, i.e. Ff_n^m is $t^{\frac{1}{p^k}}$-torsion. □

4.3 Quantitative Hebbarkeitssatz

The following proposition is an analogous to Riemann's theorem on removable singularities (*Riemannscher Hebbarkeitssatz*) in the context of perfectoid theory.

Proposition 4.3.1. *Consider a perfectoid $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$-algebra A_∞ with a system of p-power roots $\{g^{\frac{1}{p^\infty}}\}$ for some non-zero divisor element $g \in A_\infty$. Furthermore, suppose that no power of p divides g. Consider the trivial pro-module $\{A_\infty/p^m\}_n$ and the pro-module $\{A_\infty\langle\frac{p^n}{g}\rangle/p^m\}_n$ with the obvious transition maps for some fixed $m > 0$. Let $\{f_n\}_n$ be the morphism of pro-modules induced by the natural maps $A_\infty/p^m \to A_\infty\langle\frac{p^n}{g}\rangle/p^m$. Then:*

- *the modules $\ker(f_n) = 0$ for every n;*
- *the pro-module $\{\mathrm{coker}(f_n)\}_n$ is uniformly g-almost-pro-zero.*

Proof. By Lemma 3.2.8, we have the following isomorphism:

$$A_\infty \left\langle \frac{p^n}{g} \right\rangle \Big/ p^m \cong A_\infty \left[u^{\frac{1}{p^\infty}}\right] \Big/ \left(p^m, (ug)^{\frac{1}{p^\infty}} - p^{\frac{n}{p^\infty}}\right).$$

Call the rightmost module M_n: we get an induced pro-module $\{M_n\}_n$, where the transition map $t_{n+c,n} : M_{n+c} \to M_n$ is given by $u^{\frac{1}{p^l}} \mapsto (up^c)^{\frac{1}{p^l}}$ for all l.

The induced maps $A_\infty/p^m \to M_n$ are injective, so $\ker(f_n) = 0$ for all n. For the second part, we will show that given $k > 0$ there is a c such that the c-fold transition map $t_{n+c,n} : M_{n+c} \to M_n$, multiplied by $g^{\frac{1}{p^k}}$, has image in $A_\infty/p^m \subseteq M_n$. Since $t_{n+c,n}(u^{\frac{1}{p^l}}) = u^{\frac{1}{p^l}} p^{\frac{c}{p^l}}$, we have two cases:

- if $\frac{c}{p^l} \geq m$, we get that $t_{n+c,n}(u^{\frac{1}{p^l}}) \in p^m M_n = 0$;
- if $\frac{c}{p^l} < m$, $g^{\frac{1}{p^k}} t_{n+c,n}(u^{\frac{1}{p^l}}) = g^{\frac{1}{p^k}} u^{\frac{1}{p^l}} p^{\frac{c}{p^l}} = g^{\frac{1}{p^k} - \frac{1}{p^l}} (gu)^{\frac{1}{p^l}} p^{\frac{c}{p^l}} = g^{\frac{1}{p^k} - \frac{1}{p^l}} p^{\frac{n+c}{p^l}}$, which is in A_∞/p^m if $l > k$.

It is sufficient that we choose $c = p^k m$, so that $\frac{c}{p^l} < m$ if and only if $l > k$. □

Remark 4.3.2. The morphism $\{f_n\}_n$ is a pg-almost-pro-isomorphism.

Let's prove a very useful proposition in the hypotheses given to us by the statement of the previous theorem. We will assume to be working on a ring R, with an non zero divisor $t \in R$ which has a system of p^k-roots, and almost mathematics will be considered with respect to the ideal $(t^{\frac{1}{p^\infty}})$.

Proposition 4.3.3. *$\{f_n : M_n \to N_n\}_n$ be an pro-morphism of pro-R-modules such that the pro-modules $\{\ker(f_n)\}$ and $\{\mathrm{coker}(f_n)\}_n$ are both uniformly almost-pro-zero.*

Then, the natural map $\varprojlim_n M_n \to \varprojlim_n N_n$ is an almost isomorphism.

Proof. Let's call $\{p_i^j\}_{i,j}$ the transition maps for $\{M_n\}_n$ and $\{q_i^j\}_{i,j}$ the transition maps for $\{N_n\}_n$.

The induced pro-module $\{\mathrm{im}(f_n)\}_n$ allows us to split the maps f_n as $g_n \circ h_n$, where h_n is surjective and g_n is injective. Therefore, we can prove the statement in these two simpler cases.

- Suppose that $\{f_n\}_n$ is pointwise surjective.

  If $(x_n)_n \in \ker(\tilde{f})$, we have that for all i $f_i(x_i) = 0$, so $\{x_n\}_n \in \{\ker f_n\}_n$. In particular, this means that $(x_n)_n \in \varprojlim_n \ker(f_n)$, which is almost zero by Lemma 4.2.5; therefore $\tilde{f}$ is almost injective.

  Take $(y_n)_n \in \varprojlim_n N_n$. Since f_n is surjective for all n, we can consider a set $\{x_n\}_n$ such that $f_n(x_n) = y_n$. Fix some $k > 0$ and take the constant c such that for all n $t^{\frac{1}{p^k}} p_n^{n+c}$ is zero when restricted to $\ker(f_{n+c})$. We have that, for all n:

  $$f_{n+c}(p_{n+c}^{n+c+1}(x_{n+c+1})) = q_{n+c}^{n+c+1}(f_{n+c+1}(x_{n+c+1})) = q_{n+c}^{n+c+1}(y_{n+c+1}) = f_{n+c}(x_{n+c}).$$

  This means that $p_{n+c}^{n+c+1}(x_{n+c+1}) - x_{n+c}$ is contained in $\ker(f_n)$. From the definition of c we get that, if we apply to this element $t^{\frac{1}{p^k}} p_n^{n+c}(-)$ we get zero. Rewriting this equality, we get that:

  $$t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}) = t^{\frac{1}{p^k}} p_n^{n+c}(p_{n+c}^{n+c+1}(x_{n+c+1})) = t^{\frac{1}{p^k}} p_n^{n+c+1}(x_{n+c+1}).$$

  This allows us to define the sequence $(x_n^{(k)}) := (t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}))_n$. To verify that it identifies an element in $\varprojlim_n M_n$, we just need to show the following equality:

  $$p_n^{n+1}(x_{n+1}^{(k)}) = p_n^{n+1}(t^{\frac{1}{p^k}} p_{n+1}^{n+c+1}(x_{n+c+1})) = t^{\frac{1}{p^k}} p_n^{n+c+1}(x_{n+c+1}) = t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}) = x_n^{(k)},$$

  where the second to last equality follows from the previous chain of equalities. Finally:

  $$f_n(x_n^{(k)}) = f_n(t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c})) = t^{\frac{1}{p^k}} q_n^{n+c}(f_{n+c}(x_{n+c})) = t^{\frac{1}{p^k}} y_n,$$

  therefore $\tilde{f}((x_n^k)_n) = t^{\frac{1}{p^k}}(y_n)_n$. By repeating the construction for all k, we get that $\tilde{f}$ is almost surjective.

- Suppose that $\{f_n\}_n$ is pointwise injective.

  If $(x_n)_n \in \ker(\tilde{f})$, we have that for all i $f_i(x_i) = 0$, so $x_i = 0$ for all i; therefore $\tilde{f}$ is injective.

  Take $(y_n)_n \in \varprojlim_n N_n$. For all j, call $z_j \in \mathrm{coker}(f_j)$ the projection of y_j. Since $\{\mathrm{coker}(f_n)\}_n$ is almost-pro-zero, for all $k \geq 0$, for all i there is some j such that $t^{\frac{1}{p^k}} q_i^j$ is identically zero, and in particular $0 = t^{\frac{1}{p^k}} q_i^j(z_j) = t^{\frac{1}{p^k}} z_i$. This means that for all i and for all k there is

some $x_i^{(k)} \in M_i$ such that $f_i(x_i^{(k)}) = t^{\frac{1}{p^k}} y_i$, so we get:

$$f_i(p_i^{i+1}(x_{i+1}^{(k)})) = q_i^{i+1}(f_{i+1}(x_{i+1}^{(k)})) = q_i^{i+1}(t^{\frac{1}{p^k}} y_{i+1}) = t^{\frac{1}{p^k}} q_i^{i+1}(y_{i+1}) = t^{\frac{1}{p^k}} y_i = f_i(x_i^{(k)}).$$

Since f_i is injective, we get, for all i and for all k, $p_i^{i+1}(x_{i+1}^{(k)}) = x_i^{(k)}$. In particular, the sequence $(x_i^{(k)})_i$ identifies an element in $\varprojlim_n M_n$; the image of this element is $t^{\frac{1}{p^k}}(y_n)_n$, therefore, by varying k, we get that $\tilde{f}$ is almost surjective.

□

Corollary 4.3.4. *In the setting of Proposition 4.3.1, i.e. with $R := A_\infty$, $t := pg$, and the pro-R-modules $\{M_n\}_n := \{A_\infty/p^m\}_n$ and $\{N_n\}_n := \{A_\infty\langle\frac{p^n}{g}\rangle/p^m\}_n$ for some m, the hypotheses of Proposition 4.3.3 are satisfied, therefore we have:*

$$A_\infty/p^m \cong \varprojlim_n A_\infty/p^m \approx_{pg} \varprojlim_n A_\infty \left\langle \frac{p^n}{g} \right\rangle /p^m.$$

Proposition 4.3.5. *In the same setting as Proposition 4.3.3, take some R-module Q and consider the following natural map:*

$$\varprojlim_n \operatorname{Ext}^1_R(Q, M_n) \to \varprojlim_n \operatorname{Ext}^1_R(Q, N_n).$$

It is a t-almost isomorphism.

Proof. Like in the proof of Proposition 4.3.3, we just have to prove the two cases in which $\{f_n\}_n$ is respectively pointwise surjective and pointwise injective. We will only prove the first case, since the hypotheses are symmetric and the other case has a very similar proof.

Consider the pro-module $\{K_n\}_n := \{\ker(f_n)\}_n$. Applying the functor $\operatorname{Hom}_R(Q, -)$, we get the following pointwise exact sequence of pro-R-modules for all i:

$$\{\operatorname{Ext}^i_R(Q, K_n)\}_n \longrightarrow \{\operatorname{Ext}^i_R(Q, M_n)\}_n \xrightarrow{\{\phi_n\}_n} \{\operatorname{Ext}^i_R(Q, N_n)\}_n \longrightarrow \{\operatorname{Ext}^{i+1}_R(Q, K_n)\}_n.$$

Since for all i $\operatorname{Ext}^i(Q, -)$ is an R-linear functor, and $\{K_n\}_n$ is uniformly almost-pro-zero, by Proposition 4.2.6 the leftmost and rightmost pro-modules in the exact sequence are uniformly almost-pro-zero. The leftmost module admits a pointwise surjective map towards the pro-module $\{\ker(\phi_n)\}$, while the pro-module $\{\operatorname{coker}(\phi_n)\}_n$ admits a pointwise injective map towards the rightmost module. By Lemma 4.2.4 $\{\ker(\phi_n)\}$ and $\{\operatorname{coker}(\phi_n)\}_n$ are uniformly almost-pro-zero, therefore by Proposition 4.3.3 $\{\phi_n\}_n$ induces an isomorphism of the limits. □

Chapter 5

Proof of the main theorem

5.1 Properties of faithfully flat extensions

The aim of this section is to present some important results on faithful flatness.

Lemma 5.1.1. *Let A be a noetherian ring and $\{B_i\}_i$ a filtered diagram of faithfully flat A-algebras with colimit B. Then B is a faithfully flat A-algebra.*

Proof. If we drop the faithfulness condition, this lemma is true in much more generality, without the requirements of A being noetherian and B_i being algebras. For the faithfulness, take an A-module M - which we can assume to be finitely generated - and suppose that $M \otimes B = 0$: since tensor product commutes with filtered colimits, $0 = \varinjlim_i M \otimes B_i$. This implies that for any i, for any element $m_i \in M \otimes B_i$, there is $j > i$ such that the map $M \otimes B_i \to M \otimes B_j$ sends m_i to 0. In particular, for any element $m \in M$ there is some k such that the natural map $f_k : M \to M \otimes B_k$ sends m to 0. Considered the filtered system $\{\ker(f_i)\}_i$ of submodules of M: by the previous argument, their union is M, but since M is finitely generated over a noetherian ring, there is some k such that $\ker(f_k) = M$. Since $M \otimes B_k$ is generated over B_k by the elements $\{m \otimes 1 = f_k(m) | m \in M\}$, it follows that $M \otimes B_k = 0$, therefore $M = 0$ by faithful flatness. □

Lemma 5.1.2. *Let $f : A \to B$ be a faithfully flat ring map. Let C be a B-module which is flat (resp. faithfully flat) as an A-module. Then C is also flat (resp. faithfully flat) as a B-module.*

Proof. First, let's prove the flatness. Consider a generic short exact sequence of B-modules $0 \to M \to N \to P \to 0$. Since the functor $- \otimes_A C$ is the composition of the functors $- \otimes_A B$ and $- \otimes_B C$, we have the following:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\otimes_A C} & & & \searrow{\scriptstyle -\otimes_B C} & & & \\
 & & & & M \otimes_B C & \longrightarrow & N \otimes_B C & \longrightarrow & P \otimes_B C \longrightarrow 0 \\
 & & & & & & \searrow{\scriptstyle -\otimes_A B} & & \\
0 & \longrightarrow & M \otimes_A C & \longrightarrow & N \otimes_A C & \longrightarrow & P \otimes_A C & \longrightarrow & 0,
\end{array}
$$

where every sequence is exact. Since B is faithfully flat over A, and $M \otimes_A C \to N \otimes_A C$ is an injective map, so is $M \otimes_B C \to N \otimes_B C$, therefore C is a flat B-module.Moreover, suppose that C is a faithfully flat A-module and take a B-module M such that $M \otimes_B C = 0$. Then $M \otimes_A C = (M \otimes_B C) \otimes_A B = 0$, therefore $M = 0$. □

Lemma 5.1.3 ([Bha18, Proposition 5.1])**.** *Let B be a noetherian ring, and $B \to C$ a ring extension. Suppose that there is $\pi \in B$ such that the extension is (faithfully) flat modulo π, and both rings are π-adically complete and π-torsion free. Then, the extension $B \to C$ is (faithfully) flat.*

Proof. Let's prove the flatness first. We need to show that $\mathrm{Tor}_B^1(C, M) = 0$ for all B-modules M, but without loss of generality we can assume M to be finitely generated.

We will only prove this when M has no π-torsion. For the other cases, we will give just a sketch of the proof:

- if $M = (B/\pi)^k$, we consider the short exact sequence induced by the multiplication by π, $0 \to B^k \to B^k \to (B/\pi)^k \to 0$ - since C has no π torsion, it remains exact after tensoring by C, which implies that $\mathrm{Tor}_B^n(C, (B/\pi)^k) = 0$ for all $n \geq 1$ via the long exact sequence;

- if $\pi M = 0$, M is a finitely generated B/π-module, and we can consider a short exact sequence of B/π-modules $0 \to K \to (B/\pi)^k \to M \to 0$ - since C/π is flat over B/π, tensoring by C over B preserves exactness, and with a little induction on the long exact sequence we get that $\mathrm{Tor}_B^n(C, M) = 0$ for all $n \geq 1$;

- if $\pi^k M = 0$ for $k \geq 2$, we may work by induction and consider the short exact sequence $0 \to \pi M \to M \to M/\pi M \to 0$ - since the leftmost and rightmost modules are killed by π^{k-1}, they have the property by inductive hypothesis, so via long exact sequence we get $\mathrm{Tor}_B^n(C, M) = 0$;

- for a generic M, we may consider the exact sequence $0 \to K \to M \to Q \to 0$, where K is the submodule of the elements that are killed by π^k for some k (which by finite generation means that $\pi^h K = 0$ for some h), so that Q has no π-torsion - if these two particular cases have been solved, via long exact sequence we get that $\mathrm{Tor}_B^1(C, M) = 0$.

If M has no π-torsion, we have the following exact sequence:

$$0 \longrightarrow M \xrightarrow{-\cdot\pi} M \longrightarrow M/\pi M \longrightarrow 0.$$

If we tensor by C, we get the following exact sequences for all $n \geq 1$:

$$\mathrm{Tor}_B^{n+1}(C, M/\pi M) \longrightarrow \mathrm{Tor}_B^n(C, M) \xrightarrow{-\cdot\pi} \mathrm{Tor}_B^n(C, M) \longrightarrow \mathrm{Tor}_B^n(C, M/\pi M).$$

The leftmost and rightmost modules are both 0 because $M/\pi M$ is of π-torsion, therefore for all $n \geq 1$ the endomorphism of $\mathrm{Tor}_B^n(C, M)$ given by the multiplication by π is an isomorphism, i.e. $\mathrm{Tor}_B^n(C, M)$ is π-divisible.

Since M is finitely generated over a noetherian module, we may take a free resolution of M where all the free modules have finite rank:

$$\cdots \longrightarrow B^{e_{n+1}} \longrightarrow B^{e_n} \longrightarrow \cdots \longrightarrow B^{e_0} \longrightarrow M \longrightarrow 0.$$

Tensoring by C we get the following sequence:

$$\cdots \longrightarrow C^{e_{n+1}} \xrightarrow{\delta_n} C^{e_n} \xrightarrow{\delta_{n-1}} C^{e_{n-1}} \longrightarrow \cdots,$$

and by definition, for all $n \geq 1$, $\mathrm{Tor}_B^n(C, M) := \ker(\delta_{n-1})/\operatorname{im}(\delta_n)$. Since the module $\mathrm{Tor}_B^n(C, M)$ is π-divisible, for all k, for all elements $x \in \ker(\delta_{n-1})$, there is some element $y \in \ker(\delta_{n-1})$ such that $x - \pi^k y \in \operatorname{im}(\delta_n)$. We have the following containments:

$$\operatorname{im}(\delta_n) \subseteq \ker(\delta_{n-1}) = \operatorname{im}(\delta_n) + \pi^k \ker(\delta_{n-1}) \subseteq \operatorname{im}(\delta_n) + \pi^k B^{e_n},$$

which means that $\operatorname{im}(\delta_n)$ and $\ker(\delta_{n-1})$ coincide modulo π^k. Since C is π-adically complete, we have:

$$C^{e_n} \cong (\varprojlim_k C/\pi^k)^{e_n} \cong \varprojlim_k C^{e_n}/\pi^k.$$

We can write any element of C^{e_n} as $(x_k)_k$ with $x_k \in C^{e_n}/\pi^k$, and we get that:

$$\begin{aligned}(x_k)_k \in \ker(\delta_{n-1}) &\Leftrightarrow \forall k\ x_k \in \ker(\delta_{n-1})/(\pi^k C \cap \ker(\delta_{n-1})) \\ &\Leftrightarrow \forall k\ x_k \in \operatorname{im}(\delta_n)/(\pi^k C \cap \ker(\delta_n)) \\ &\Leftrightarrow (x_k)_k \in \operatorname{im}(\delta_n),\end{aligned}$$

therefore the two coincide as submodules of C^{e_n} and $\mathrm{Tor}_B^n(C, M) = 0$.

Let's now prove that, if $B/\pi \to C/\pi$ is faithfully flat, so is $B \to C$.

We already know that C is flat over B by the previous point, so we need to show that for all B-modules M, if $C \otimes_B M = 0$ then $M = 0$. With some standard finiteness arguments, we can suppose M to be finitely generated. Since C/π is faithfully flat over B/π and $C/\pi \otimes_{B/\pi} M/\pi = 0$, we have that $M/\pi M = 0$, i.e. $M = \pi M$. Since M is finitely generated, by Nakayama's lemma, if π belongs to the Jacobson radical of A, $M = 0$. We just need to show that $\pi \in \mathfrak{p}$ for every maximal ideal $\mathfrak{p} \subseteq A$. Fix a maximal $\mathfrak{p}$ and suppose by contradiction that $\pi \notin \mathfrak{p}$: by maximality, for all k, $\mathfrak{p} + \pi^k B = B$. This means that $\mathfrak{p}$ coincides with B modulo π, so with a similar argument as the previous point, we get that $\mathfrak{p} = B$, which is a contradiction. □

Corollary 5.1.4. *Let $\{B_k, \alpha_{i,j}\}_{i,j,k}$ a filtered diagram of A_0-algebras with no p-torsion, and let B_∞ be the p-adic completion of their colimit. If $A_0/p \to B_k/p$ is faithfully flat for all k, so is the map $A_0 \to B_\infty$.*

Proof. Since for all i, j $\alpha_{i,j}(pB_i) \subseteq pB_j$, we can work modulo p: in this context, the p-adic completion becomes a trivial functor, so $B_\infty/p = \varinjlim_k B_k/p$. Since A_0/p is noetherian, filtered colimits preserve faithful flatness by Lemma 5.1.1, therefore $A_0/p \to B_\infty/p$ is faithfully flat. For all k, the map $B_k \xrightarrow{\cdot p} B_k$ is injective because B_k has no p-torsion; since filtered colimits are exact, multiplication by p is injective on $B_{\infty,0} := \varinjlim_k B_k$. We conclude by proving that the p-adic completion of a ring with no p-torsion does not have p-torsion: if $x := \{x_i | x_i \in B_{\infty,0}/p^i\}_i$ is a p-torsion element in B_∞, for all n one must have $px_n = 0$, i.e. $p\tilde{x_n} = p^n b$ for some element $b \in B_{\infty,0}$, where $\tilde{x_n} \in B_{\infty,0}$ is a lifting of x_n; therefore $\tilde{x_n} = p^{n-1}c$ because $B_{\infty,0}$ has no p-torsion, which means that $x_{n-1} = 0$, therefore $x = 0$. Since the hypotheses of Lemma 5.1.3 are satisfied, the map $A_0 \to B_\infty$ is faithfully flat. □

5.2 Preliminary constructions

Throughout the rest of the chapter, we will assume A_0 to have all the properties of the second point of Proposition 1.1.8. The first thing that we want to do is to take the problem into the realm of perfectoid theory. We need a suitably "faithful" extension of the ring A_0, which will also be a perfectoid algebra. For the construction we will need the following structure theorem for regular rings of mixed characteristic.

Theorem 5.2.1 (Cohen)**.** *Let $(A, \mathfrak{m})$ be a complete regular local ring of mixed characteristic $(0, p)$ and dimension d, and let k be its residue field, which we assume to be perfect. Then:*

- *if $p \in \mathfrak{m} \setminus \mathfrak{m}^2$, there is an isomorphism $W(k)[[x_1, \cdots, x_{d-1}]] \cong A$;*
- *if $p \in \mathfrak{m}^2$, there is a surjective map $W(k)[[x_1, \cdots, x_d]] \twoheadrightarrow A$;*

where $W(k)$ is the ring of Witt vectors of k.

Before proceeding, let's give an additional definition.

Definition 5.2.2. Consider an extension $A_0 \to A$, where A admits a system of p-power roots of p. The ring A is said to be *faithfully almost flat* over A if the following conditions are satisfied. For all $M \in A - \mathbf{Mod}$:

- as an A-module, $\mathrm{Tor}_A^1(M, A)$ is p-almost zero;
- if $M \otimes_{A_0} A \approx_p 0$, then $M = 0$.

Remark 5.2.3. The first condition is a weakened version of flatness, hence the term almost flat. On the other hand, the second condition is a priori stronger that faithfulness - it can be proven that it is not (see [Bha14, Lemma 2.1]), but it won't be needed in this thesis.

Remark 5.2.4. Faithful almost flatness is well defined in the following sense: if $A_0 \to A$ is faithfully almost flat and $A \to A'$ is an almost faithfully flat, then $A_0 \to A'$ is also faithfully almost flat.

Proposition 5.2.5 ([Bha18, Proposition 5.2])**.** *There is an extension $A_0 \to A_{\infty,0}$ such that $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ is a perfectoid affinoid $\mathbb{Q}_p(\widehat{p^{\frac{1}{p^\infty}}})$-algebra, and that $A_{\infty,0}$ is faithfully almost flat over A_0. Furthermore, $A_{\infty,0}$ is also flat over A_0.*

Proof. Since we are in the hypotheses of Proposition 1.1.8, Cohen's structure theorem tells us that A_0 is isomorphic to the ring of power series $W[[x_1, \cdots, x_{d-1}]]$, where $d := \dim(A_0)$ and W is the ring of Witt vectors associated to the residue field of A_0 (therefore W is a complete discrete valuation domain, with maximal ideal generated by p).

Define $A_n := W[[p^{\frac{1}{p^n}}, x_1^{\frac{1}{p^n}}, \cdots, x_{d-1}^{\frac{1}{p^n}}]]$, and call $A_{\infty,0}$ the p-adic completion of their filtered colimit. For all m the natural inclusion $A_0/p \hookrightarrow A_m/p$ is free, therefore faithfully flat; by Corollary 5.1.4, the map $A_0 \to A_{\infty,0}$ is faithfully flat.

Let's take an A_0-module M such that $M \otimes A_{\infty,0} \approx 0$, and prove that $M = 0$. We can assume M to be generated by one element, i.e. $M = A_0/I$ for some ideal $I \subseteq A_0$.

Faithful flatness tells us that $I_{\infty,0} := I \otimes A_{\infty,0}$ is an ideal of $A_{\infty,0}$. We have the almost isomorphisms $0 \approx M \otimes A_{\infty,0} \cong A_{\infty,0}/I_{\infty,0}$, so for every k we have $p^{\frac{1}{p^k}} \in I_{\infty,0}$, or alternatively $p \in (I_{\infty,0})^{p^k}$.

The inclusion $I^{p^k} \subseteq A_0$ induces an injective map $i_k : I^{p^k} \otimes A_{\infty,0} \to A_{\infty,0}$, whose image is contained in $(I_{\infty,0})^{p^k}$. Using the natural surjection $\bigotimes_1^{p^k} I \twoheadrightarrow I^{p^k}$, we get this commutative diagram:

$$\begin{array}{ccccc} (I \otimes_{A_0} \cdots \otimes_{A_0} I) \otimes_{A_0} A_{\infty,0} & \longrightarrow & I^{p^k} \otimes_{A_0} A_{\infty,0} & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow i_k & & \\ I_{\infty,0} \otimes_{A_{\infty,0}} \cdots \otimes_{A_{\infty,0}} I_{\infty,0} & \longrightarrow & (I_{\infty,0})^{p^k} & \longrightarrow & 0, \end{array}$$

so i_k is surjective, and therefore bijective. With a similar line of reasoning, one gets that the ideal $(I_{\infty,0})^{p^k} \cong (I_{\infty,0})^{p^k} + (p)_{A_{\infty,0}} \subseteq A_{\infty,0}$ is almost isomorphic to $(I^{p^k} + (p)_{A_0}) \otimes_{A_0} A_{\infty,0}$ via the natural map. Since $A_0 \to A_{\infty,0}$ is faithfully flat, we get that $I^{p^k} + (p)_{A_0} = I^{p^k}$, therefore $p \in I^{p^k}$ for all k. By hypothesis, $p \notin \mathfrak{m}^2$, so the only possibility is that $I = A_0$, i.e. $M = 0$.

Finally, to show that $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ is an integral perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, it suffices to check the following.

- The ring $A_{\infty,0}[\frac{1}{p}]$ is a $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, p-adically complete by definition.
- We have an equality $A_{\infty,0} = A_{\infty,0}[\frac{1}{p}]^\circ$ (essentially by definition) and the Frobenius endomorphism Φ on $A_{\infty,0}/p$ is surjective because $A_{\infty,0}/p = \bigcup_m A_m/p$, and $\Phi(A_{m+1}/p) = A_m/p$ for all m.
- The ring $A_{\infty,0}$ is integral. If $ab = 0$ in $A_{\infty,0}$, modulo p^{2n} $\bar{a}, \bar{b} \in A_m/p^{2n}$ for some m, so they have representatives $\tilde{a}, \tilde{b} \in A_m$. Since $p^{\frac{1}{p^m}} A_m$ is a prime ideal and $\tilde{a}\tilde{b} \in (p^{\frac{1}{p^m}})^{2np^m}$, one of them must be contained in $(p^{\frac{1}{p^m}})^{np^m} = (p^n)$; in particular, either $\bar{a}$ or $\bar{b}$ is divisible by p^n. Repeating the argument for all n we get that either a or b is in (p^n) for all n, i.e. it is zero.
- The ring $A_{\infty,0}$ is normal. Take a monic polynomial $f(x) \in A_{\infty,0}[x]$ and suppose it has a root $\frac{b}{a}$ in the fraction field. For all n, we can approximate any $x \in A_{\infty,0}$ with an element $x_n \in \varinjlim_m A_m$ such that they are equal modulo p^n. If we call $\tilde{f}_n$ a monic polynomial whose coefficients approximate those of f in the same fashion, $c_n := \tilde{f}_n(\frac{b_n}{a_n}) \in \varinjlim_m A_m$ is zero modulo p^n: we now define $f_n := \tilde{f}_n - c_n$, as another monic approximation of f, such that $f_n(\frac{a_n}{b_n}) = 0$. The coefficients of f_n, a_n, and b_n are all contained in A_m for some big enough m; but A_m is a power series ring over a discrete valuation domain, so it is a unique factorization domain, and in particular normal: $\frac{a_n}{b_n} \in A_m \subseteq \varinjlim_m A_m$. By varying n, we get an approximating sequence $\left\{\frac{a_n}{b_n}\right\}_n$ of $\frac{a}{b}$: since all the elements in the sequence are in $\varinjlim_m A_m$, their limit is contained in $A_{\infty,0}$.

□

We follow with another result which allows us to enrich the extension $A_{\infty,0}$ with a set system of p-power roots.

Theorem 5.2.6 (André)**.** *Let $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ be the perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra defined above, and $g \in A_0$ coprime with p. It admits an integral perfectoid extension $A_{\infty,0} \to A_\infty$ with a system of p-power roots of g, such that $A_0 \to A_\infty$ is faithfully almost flat.*

Let's first discuss the geometric idea behind this proof. We can take $A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle$ (i.e. the ring obtained adjoining a p-power system of roots of T to $A_{\infty,0}$ and taking the p-adic completion), and work with the pair $(A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle[\frac{1}{p}], A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle)$: it is a perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, and we call Y its associated space. Intuitively, we want to impose $T = g$, which geometrically would correspond to considering the following subset:

$$\{y \in Y || (T-g)(y)| = 0\} \subset Y.$$

To do this, we write this subset as intersection of the open sets

$$U_n := U\left(\frac{T-g, p^n}{p^n}\right) = \{y \in Y || (T-g)(y)| \leq |(p^n)(y)| = |p|^n\},$$

and we take the colimit of $\mathcal{O}_Y^+(U_n)$: its p-adic completion will be the ring A_∞ (we will omit the proof that A_∞ is almost isomorphic to an integral perfectoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, but this result can be found in [Sch13, Chapter II.2]). As p^n divides $T-g$ in $\mathcal{O}_Y^+(U_n)$, in A_∞ the image of $T-g$ is divisible by every power of p, therefore it is zero because of completeness.

To explicitly describe the ring $\mathcal{O}_Y^+(U_n)$, we resort to the approximation lemma: by Proposition 3.2.7 we can find $f_n \in A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle^\flat$ such that the rational subsets $U(\frac{T-g,p^n}{p^n})$ and $U(\frac{f_n^\#,p^n}{p^n})$ are the same, and $f_n^\# \equiv T-g \mod p^{\frac{1}{p}}$. Crucially, $f_n^\#$ has a system of p-power roots, and by Lemma 3.2.9 we could describe the ring at the almost level as:

$$\mathcal{O}_Y^+(U_n) \approx A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle \left\langle \left(\frac{f_n^\#}{p^n}\right)^{\frac{1}{p^\infty}}\right\rangle,$$

which is the p-adic completion of $\varinjlim_k C_{n,k}$, where:

$$C_{n,k} := A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle \left[u_n^{\frac{1}{p^\infty}}\right] \Big/ \left((u_n p^n)^{\frac{1}{p^k}} - f_n^{\#\frac{1}{p^k}}\right).$$

In the following proof, we will be able to bypass much of the perfectoid theory that we took for granted in this prelude, only using the existence of $f_n^\#$ as described above, as we will simply show that the completion of the colimit of the rings $C_{n,k}$, which admits a system of p-power roots for g, is a faithfully flat extension of $A_{\infty,0}$.

Proof. If we work modulo p, we need not worry about completions, and we can describe A_∞/p at the almost level as $\varinjlim_{n,k} C_{n,k}/p$, where the maps $\varinjlim_k C_{n,k} \to \varinjlim_k C_{n+1,k}$ exist because one can check that if $p^{\frac{n+1}{p^k}} | f_{n+1}^{\#\frac{1}{p^k}}$ then $p^{\frac{n}{p^k}} | f_n^{\#\frac{1}{p^k}}$: this follows from the approximation condition on $f_n^\#$ and $f_{n+1}^\#$.

Let's prove first that $A_{\infty,0} \to C_{n,k}$ is faithfully flat modulo p^{ε_k}, with $\varepsilon_k = \frac{1}{p^{k+1}}$:

$$C_{n,k}/p^{\varepsilon_k} = A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle \left[u_n^{\frac{1}{p^\infty}}\right] \Big/ \left(p^{\varepsilon_k}, f_n^{\#p^{\varepsilon_k}}\right).$$

The k-fold Frobenius identifies $C_{n,k}/p^{\varepsilon_k}$, as an $A_{\infty,0}/p^{\varepsilon_k}$-algebra, with the $A_{\infty,0}/p^{\frac{1}{p}}$-algebra:

$$A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle \left[u_n^{\frac{1}{p^\infty}}\right] \Big/ \left(p^{\frac{1}{p}}, f_n^\#\right) \cong A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle \left[u_n^{\frac{1}{p^\infty}}\right] \Big/ \left(p^{\frac{1}{p}}, T-g\right),$$

where we used that $f_n^{\#} \equiv T - g \mod p^{\frac{1}{p}}$. This algebra is free over $A_{\infty,0}/p^{\frac{1}{p}}$, and in particular faithfully flat. Furthermore, since no power of p divides $f_n^{\#}$, $C_{n,k}$ has no p-torsion.

Recall that $A_m \to A_{\infty,0}$ is faithfully flat, where A_m is defined as in 5.2.5. The following maps are faithfully flat for all n, k:

- the maps $A_m/p^{\varepsilon_k} \to A_{\infty,0}/p^{\varepsilon_k}$, being the quotient of $A_m \to A_{\infty,0}$;
- the maps $A_m/p^{\varepsilon_k} \to C_{n,k}/p^{\varepsilon_k}$, by composition with $A_{\infty,0}/p^{\varepsilon_k} \to C_{n,k}/p^{\varepsilon_k}$;
- the maps $A_m \to \widehat{C_{n,k}}$, by Lemma 5.1.3, since A_k is noetherian, p^{ε_k}-adically complete and with no p^{ε_k}-torsion, while $\widehat{C_{n,k}}$ is p^{ε_k}-adically complete by definition and has no p-torsion because $C_{n,k}$ has no p^{ε_k}-torsion (as seen in the proof of Corollary 5.1.4);
- the maps $A_0 \to \widehat{C_{n,k}}$, by precomposition with $A_0 \to A_k$;
- the maps $A_0/p \to C_{n,k}/p$, obtained by the previous one modulo p;
- the map $A_0 \to A_\infty$, by Corollary 5.1.4.

Let's remark that if we work with A_∞ only up to almost isomorphism (since it is only almost isomorphic to an integral perfectoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra), the map $A_0 \to A_\infty$ has to be considered just almost faithfully flat. □

Remark 5.2.7. Via Lemma 5.1.2, it's also possible to prove that $A_{\infty,0} \to A_\infty$ is almost faithfully flat.

Let's include an easy lemma:

Lemma 5.2.8. *Let A_0 be a local noetherian ring of characteristic $(0, p)$ with no p-torsion. Then for all finitely generated A_0-modules M the natural map*

$$\mathrm{Ext}^1_{A_0}(M, A_0) \to \varprojlim_k \mathrm{Ext}^1_{A_0}(M, A_0/p^k)$$

is injective.

Proof. Since A_0 is noetherian, M is finitely presented. It is thus possible to find a free resolution:

$$\cdots \to A_0^n \to A_0^m \to M \to 0,$$

with m, n finite. As a consequence, $\mathrm{Ext}^1_{A_0}(M, A_0)$ is finitely generated, since it is the quotient of a submodule of $\mathrm{Hom}_{A_0}(A_0^n, A_0) \cong A_0^n$. Since A_0 has no p-torsion, we get the following short exact sequence:

$$0 \longrightarrow A_0 \xrightarrow{\cdot p^k} A_0 \longrightarrow A_0/p^k \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{A_0}(M, -)$, we obtain the following long exact sequence:

$$\cdots \longrightarrow \mathrm{Ext}^1_{A_0}(M, A_0) \xrightarrow{\cdot p^k} \mathrm{Ext}^1_{A_0}(M, A_0) \xrightarrow{f_m} \mathrm{Ext}^1_{A_0}(M, A_0/p^k) \longrightarrow \cdots.$$

In particular, $\ker(f_m) = p^k \mathrm{Ext}^1_{A_0}(M, A_0)$, therefore the limit of the maps $\{f_m\}_m$ has kernel $K := \bigcap p^k \mathrm{Ext}^1_{A_0}(M, A_0)$. Since $K = pK$, by Nakayama's lemma we get that $K = 0$. □

5.3 Proof

Now we are ready to face the main theorem:

Theorem 5.3.1 (André)**.** *Let A_0 be a local and complete regular ring of characteristic $(0,p)$, and $i : A_0 \hookrightarrow B_0$ a module-finite extension of rings. The inclusion i splits as a map of A_0-modules.*

First, let's give an idea of the proof. By Proposition 2.4.10 there is some $g \in A_0$ such that $A_0[\frac{1}{g}] \hookrightarrow B_0[\frac{1}{g}]$ is finite étale. Since étaleness is preserved by base change, $A_\infty[\frac{1}{g}] \to B_0 \otimes_{A_0} A_\infty[\frac{1}{g}]$ is still étale. We want to push all the ramification of the extension $A_\infty \to B_0 \otimes_{A_0} A_\infty$ in p by adjoining elements to A_∞ such that g divides p in this new ring: in this way the hypothesis of Faltings'almost purity theorem will be satisfied. On one hand, perfectoid theory allows us to find such an extension; on the other hand, this extension won't be almost faithfully flat: this means that we will not be able to deduce from the almost-splitting of the tensored sequence the almost-splitting of the starting sequence. This problem will be solved by the Hebbarkeitssatz theorem, which allows us to reduce modulo p^m at the cost of shifting from p-almost mathematics to pg-almost mathematics.

Proof. By Proposition 2.4.10, there is an element $g' \in A_0$ such that $A_0[g'^{-1}] \to B_0[g'^{-1}]$ is an étale covering. Since A_0 is a regular local ring, it is a unique factorization domain, therefore there is some k such that $g' = p^k g$ for some g coprime with p (this condition of coprimality will be necessary later to apply Proposition 4.3.1)

As already seen in the first chapter, the sequence $0 \to A_0 \to B_0 \to Q_0 \to 0$ splits if and only if $\alpha_0 \in \mathrm{Ext}^1_{A_0}(Q_0, A_0)$ is 0. Since B_0 is finitely generated as an A_0-module, so is Q_0; we can apply Lemma 5.2.8 to get the following immersion:

$$\mathrm{Ext}^1(Q_0, A_0) \hookrightarrow \varprojlim_m \mathrm{Ext}^1(Q_0, A_0/p^m).$$

We can write α_0 as the limit of $\{\alpha_0/p^m\}_m$, where α_0/p^m is the image of α_0 via the map $\mathrm{Ext}^1(Q_0, A_0) \to \mathrm{Ext}^1(Q_0, A_0/p^m)$ induced by the projection. In particular, we just need to show that $\alpha_0/p^m = 0$ for a large enough m. Let's call α_∞ and α_∞/p^m the images respectively of α_0 and α_0/p^m via the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ext}^1_{A_0}(Q_0, A_0) & \longrightarrow & \mathrm{Ext}^1_{A_0}(Q_0, A_\infty) & \xrightarrow{\approx_p} & \mathrm{Ext}^1_{A_\infty}(Q_0 \otimes A_\infty, A_\infty) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_{A_0}(Q_0, A_0/p^m) & \longrightarrow & \mathrm{Ext}^1_{A_0}(Q_0, A_\infty/p^m) & \xrightarrow{\approx_p} & \mathrm{Ext}^1_{A_\infty}(Q_0 \otimes A_\infty, A_\infty/p^m),
\end{array}$$

where the rightmost maps are p-almost isomorphisms because A_∞ is p-almost isomorphic to a flat A_0-module.

Suppose that $\alpha_0/p^m \neq 0$, which is equivalent to saying that $I := \mathrm{Ann}_{A_0}(\alpha_0/p^m) \subsetneq A_0$. Since A_0 is noetherian and I is a proper ideal, $\bigcap_n I^n = 0$ so, for a large enough k, $pg \notin I^{p^k}$, or equivalently $\frac{I^{p^k} + (pg)_{A_0}}{I^{p^k}} \neq 0$. Let $I_\infty := \mathrm{Ann}_{A_\infty}(\alpha_\infty/p^m)$. Since $A_0 \to A_\infty$ is faithfully almost flat (as in Definition 5.2.2), we have:

$$\begin{array}{ccccccccc}
 & & I \otimes A_\infty & \xrightarrow{i} & A_\infty & \longrightarrow & \langle \alpha_0/p^m \rangle_{A_0} \otimes A_\infty & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \| & & \downarrow & & \\
0 & \longrightarrow & I_\infty & \longrightarrow & A_\infty & \longrightarrow & \langle \alpha_\infty/p^m \rangle_{A_\infty} & \longrightarrow & 0,
\end{array}$$

where i is p-almost injective, which means that π is not only surjective, but also p-almost injective - in particular, it is a p-almost isomorphism: $I \otimes A_\infty \approx_p I_\infty$. In a similar fashion, we can obtain that $(pg)_{A_0} \otimes_{A_0} A_\infty \approx_p (pg)_{A_\infty}$.

Using the natural surjection $\bigotimes_1^{p^k} I \twoheadrightarrow I^{p^k}$, we get this commutative diagram:

$$\begin{array}{ccccc} (I \otimes_{A_0} \cdots \otimes_{A_0} I) \otimes_{A_0} A_\infty & \longrightarrow & I^{p^k} \otimes_{A_0} A_\infty & \longrightarrow & 0 \\ \downarrow \approx_p & & \downarrow \pi_{p^k} & & \\ I_\infty \otimes_{A_\infty} \cdots \otimes_{A_\infty} I_\infty & \longrightarrow & (I_\infty)^{p^k} & \longrightarrow & 0, \end{array}$$

so π_{p^k} is itself an almost isomorphism (for the sake of accuracy, it is surjective and almost injective). With a similar line of reasoning, one gets that the ideal $(I_\infty)^{p^k} + (pg)_{A_\infty} \subseteq A_\infty$ is almost isomorphic to $(I^{p^k} + (pg)_{A_0}) \otimes_{A_0} A_\infty$ via the natural map. Since $A_0 \to A_\infty$ is faithfully almost flat, and $\frac{I^{p^k}+(pg)_{A_0}}{I^{p^k}} \neq 0$, we get:

$$0 \not\approx_p \frac{I^{p^k} + (pg)_{A_0}}{I^{p^k}} \otimes_{A_0} A_\infty \approx_p \frac{(I^{p^k} + (pg)_{A_0}) \otimes_{A_0} A_\infty}{I^{p^k} \otimes_{A_0} A_\infty} \approx_p \frac{(I_\infty)^{p^k} + (pg)_{A_\infty}}{(I_\infty)^{p^k}}.$$

In particular, the last module is not 0, which means that $pg \notin (I_\infty)^{p^k}$. Since A_∞ has a system of p-power roots for both p and g, it means that $(pg)^{\frac{1}{p^k}} \notin I_\infty = \mathrm{Ann}_{A_\infty}(\alpha_\infty/p^m)$. Recapping, we have proven that if $\alpha/p^m \neq 0$, then α_∞/p^m is not killed by all the p-power roots of pg: to conclude the theorem we must come to a contradiction by showing that indeed α_∞/p^m is pg-almost zero.

Let $A_\infty^{(n)} := A_\infty\langle\frac{p^n}{g}\rangle$, $B_\infty^{(n)} := B_0 \otimes_{A_0} A_\infty^{(n)}$, and call their cokernel $Q_\infty^{(n)}$. Similarly, in the extension $A_\infty \to B_0 \otimes_{A_0} A_\infty$, call the second ring B_∞ and the cokernel of the extension Q_∞.

The natural map $A_\infty^{(n)} \to B_\infty^{(n)}$ becomes an étale extension after inverting p: since g divides p^n in $A_\infty^{(n)}$, inverting p also inverts g and thus kills all algebraic obstructions to étaleness (including injectivity, by virtue of faithful flatness).

The inclusion $A_\infty^{(n)}[\frac{1}{p}] \hookrightarrow B_\infty^{(n)}[\frac{1}{p}]$ is a finite étale covering, and the first ring is a perfectoid $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$-algebra, so by Theorem 3.1.21 $A_\infty^{(n)} \to B_\infty^{(n)}$ is a finite étale covering in the category $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}_a - perf$. Consider the p-almost exact sequence:

$$0 \longrightarrow A_\infty \longrightarrow B_\infty \longrightarrow Q_\infty \longrightarrow 0.$$

If we tensor by $A_\infty^{(n)}$ over A_∞ we get the p-almost exact sequence:

$$A_\infty^{(n)} \longrightarrow B_\infty^{(n)} \longrightarrow Q_\infty^{(n)} \longrightarrow 0,$$

where, being a p-almost étale covering, the leftmost map is also p-almost injective. Applying the functor $\mathrm{Hom}_{A_\infty}(-, A_\infty^{(n)})$ to the first sequence and the functor $\mathrm{Hom}_{A_\infty^{(n)}}(-, A_\infty^{(n)})$ to the second

sequence, we can consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{A_\infty}(B_\infty, A_\infty^{(n)}) & \longrightarrow & \mathrm{Hom}_{A_\infty}(A_\infty, A_\infty^{(n)}) & \longrightarrow & \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}) & \longrightarrow & \cdots \\
\uparrow & & \uparrow\cong & & \hat{\uparrow} j & & \\
\mathrm{Hom}_{A_\infty^{(n)}}(B_\infty^{(n)}, A_\infty^{(n)}) & \longrightarrow & \mathrm{Hom}_{A_\infty^{(n)}}(A_\infty^{(n)}, A_\infty^{(n)}) & \longrightarrow & \mathrm{Ext}^1_{A_\infty^{(n)}}(Q_\infty^{(n)}, A_\infty^{(n)}) & \longrightarrow & 0,
\end{array}$$

where the horizontal maps are p-almost exact, and the vertical maps are induced by the natural map $A_\infty \to A_\infty^{(n)}$. For the p-almost surjectivity of the rightmost map on the bottom, we used that $\mathrm{Ext}^1_{A_\infty^{(n)}}(B_\infty^{(n)}, A_\infty^{(n)}) \approx_p 0$ because $B_\infty^{(n)}$ is p-almost projective as an $A_\infty^{(n)}$-module; This induces the p-almost injective map j. Let $\alpha_\infty^{(n)} \in \mathrm{Ext}^1_{A_\infty^{(n)}}(Q_\infty^{(n)}, A_\infty^{(n)})$ be the element which represents the extension $A_\infty^{(n)} \to B_\infty^{(n)}$, in the same way as α_0 represents the extension $A_0 \to B_0$: via the p-almost injective map j, we can identify it with an element of $\mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)})$, which maps to $\alpha_\infty^{(n)}/p^m \in \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m)$ via the obvious map. Since $A_\infty^{(n)} \to B_\infty^{(n)}$ is a p-almost étale covering, $Q_\infty^{(n)}$ is p-almost projective as an $A_\infty^{(n)}$-module, which means that $\alpha_\infty^{(n)}$ is p-almost zero, so $\alpha_\infty^{(n)}/p^m$ is p-almost zero for every m.

For every m we have the following canonical maps. First:

$$\mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty/p^m) \to \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m),$$

which sends α_∞/p^m to $\alpha_\infty^{(n)}/p^m \approx_p 0$; passing to the limit we get the other map:

$$f : \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty/p^m) \to \varprojlim_n \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m),$$

which sends α_∞/p^m to a p-almost zero element.

By Proposition 4.3.1, the pro-morphism from $\{A_\infty/p^m\}_n$ to $\{A_\infty^{(n)}/p^m\}_n$ has kernel and cokernel uniformly pg-almost-pro-zero, so by Proposition 4.3.5 the map f is a pg-almost isomorphism: in particular, $\alpha_\infty/p^m \approx_{pg} 0$, which concludes the proof. □

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